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Tachyonic ionization cross sections of hydrogenic systems

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Abstract

Transition rates for induced and spontaneous tachyon radiation in hydrogenic systems as well as the transversal and longitudinal ionization cross sections are derived. We investigate the interaction of the superluminal radiation field with matter in atomic bound–bound and bound–free transitions. Estimates are given for Ly-α transitions effected by superluminal quanta in hydrogen-like ions. The tachyonic photoelectric effect is scrutinized, in the Born approximation and at the ionization threshold. The angular maxima occur at different scattering angles in the transversal and longitudinal cross sections, which can be used to sift out longitudinal tachyonic quanta in a photon flux. We calculate the tachyonic ionization and recombination cross sections for Rydberg states and study their asymptotic scaling with respect to the principal quantum number. At the ionization threshold of highly excited states of order $n \sim 10^4$, the longitudinal cross section starts to compete with photoionization, in recombination even at lower levels.

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1. Introduction

When considering superluminal quanta, we may try a wave theory or a particle picture as the starting point. The latter has been studied for quite some time, but did not result in viable interactions with matter [1–7]. Here, tachyons are modelled as wave fields with negative mass-squared, coupled by minimal substitution to subluminal particles. Interaction with matter is indeed the crucial point, we maintain the best established interaction mechanism, minimal substitution, by treating tachyons like photons with negative mass-squared, a real Proca field minimally coupled to subluminal matter [8–10].

The superluminal energy flux can be split into a transversal and a longitudinal component, and the different polarizations are quantized in different statistics to obtain a positive definite energy operator; transversal quanta are bosonic and longitudinal ones are fermionic. The spin-statistics theorem and most other quantum field theoretic no-go theorems are not applicable outside the lightcone, as they are based on microcausality, which means, in a relativistic context, on the non-existence of superluminal signal transfer [11]. This theory of superluminal radiation is non-relativistic, based on an absolute cosmic spacetime conception, which is crucial to maintain causality [12–14]. The superluminal modes are coupled to a Klein–Gordon field, a scalar subluminal particle in a Coulomb potential. We derive the T-matrix elements of the interaction operator, in effect, the tachyonic Einstein coefficients. The minimally coupled radiation field is treated perturbatively in linear order, as suggested by the very small tachyonic interaction constant, the ratio of tachyonic and electric fine structure constants being $\alpha_q/\alpha_e \approx 1.4 \times 10^{-11}$. The electromagnetic second-order contribution overpowers the tachyonic counterpart by some 22 orders, so that elementary statistical procedures such as detailed balancing are sufficient for the second quantization of the interaction. Linearization on account of the tiny interaction is used throughout; there is no need to develop a perturbation theory beyond the linear order. One can also reckon that the Lagrangian of the Proca field is itself just the linearization of a nonlinear Born–Infeld type of Lagrangian, as this seems to be the most straightforward way to achieve a finite classical self-energy.

This paper is about the interaction of superluminal radiation modes with energy levels of hydrogen-like ions. The emphasis is on the actual transition rates and cross sections, bound–bound and bound–free, in both directions, ionization and recombination. In which energy range can superluminal and in particular longitudinal radiation emerge? Is the longitudinal component completely overpowered by transversal electromagnetic radiation? Not so, we give quantitative estimates in this regard.

In section 2, we derive the transition rates for bound–bound transitions in hydrogen-like ions, effected by longitudinal and transversal superluminal quanta. This, after having outlined the coupling of the tachyonic radiation field to a relativistic spinless charge in a Coulomb potential and the set-up of second quantization, to keep the paper self-contained. We elaborate on transition rates in dipole approximation, on the Einstein A- and B-coefficients, on spontaneous and induced radiation outside the lightcone, and we compare tachyonic Ly-α transitions to electromagnetic ones.

The main emphasis is on tachyonic ionization; in section 3 we discuss ground state ionization. The interaction of tachyons with low energy particles can result in very speedy superluminal quanta, in contrast to high energy interactions, where the energy transfer is at best moderate so that the superluminal velocities are always close to the speed of light. The focus, therefore, is not the Born approximation, which requires energies far higher than the ionization threshold, but we rather investigate the threshold itself in dipole approximation. Extremely low energy transfer, with electronic ejection energies close to zero, is also the topic of section 4, where we study ionization of Rydberg states. The differential cross sections can be used to separate transversal and longitudinal radiation. The peaks of the transversal and longitudinal cross sections are located at noticeably different scattering angles, the transversal maximum coincides with the longitudinal minimum.

In section 4, we discuss the Born and dipole approximations of the ionization cross sections of Rydberg states. Bound states with quantum numbers $n \sim 100$ and beyond are of particular interest with regard to tachyonic ionization and recombination, as their ionization energy is singularly small, which means tiny energy transfer and high superluminal velocities if the energy of the free electron is close to zero. More importantly, the n-scaling of the longitudinal cross section at the ionization threshold is key to overcoming the very small ratio of tachyonic and electric fine structure constants in search for the longitudinal radiation. We restrict ourselves to s-states, though there is no real obstacle to considering non-zero angular momentum. The ionization cross sections depend on two asymptotic parameters, the quantum number n of the s-state and the kinetic energy of the ejected electron. The wavefunctions of hydrogenic Rydberg states are essentially Laguerre polynomials of order n, and the matrix elements in the cross sections are composed of hypergeometric polynomials of the same order. Approximations of these polynomials are calculated in the appendix. In section 5, we present our conclusions.

2. Tachyonic emission and absorption rates in hydrogenic systems

We consider a subluminal, spinless quantum particle, coupled by minimal substitution to the tachyonic vector potential: $L = L_{\mathrm{P}} + L_{\psi}$, where $L_{\mathrm{P}} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}m_{\mathrm{t}}^2 A_{\alpha}A^{\alpha}$ is the Lagrangian of the superluminal Proca field, and L_{ψ} is the Klein–Gordon Lagrangian including the interaction terms,

$$\begin{aligned} &L_{\psi} := c^{-2}\partial_t^A\psi\,\partial_t^{A*}\psi^* - \nabla^A\psi\nabla^{A*}\psi^* - (mc/\hbar)^2\psi\psi^*, \\ &\partial_t^A := \partial_t + \mathrm{i}\hbar^{-1}V - \mathrm{i}\tilde{q}A_0, \qquad \nabla^A := \nabla - \mathrm{i}\tilde{q}\mathbf{A}, \qquad V := -Ze^2/(4\pi r). \end{aligned} \tag{2.1}$$

The tachyon mass m_{t} has the dimension of an inverse length, a shortcut for $m_{\mathrm{t}}c/\hbar$. In L_{ψ}, we have included a time-independent potential V, exemplified as attractive Coulomb potential, which is treated non-perturbatively. The expansion is in $\tilde{q} := q/(\hbar c)$, where q is the tachyonic charge of the subluminal particle by which it couples to the superluminal radiation field. The Hamiltonian corresponding to (2.1) reads

$$H_{\psi} = c^2\pi\pi^* + \mathrm{i}(\tilde{q}A_0 - \hbar^{-1}V)\psi\pi - \mathrm{i}(\tilde{q}A_0 - \hbar^{-1}V)\psi^*\pi^* + \nabla^A\psi\nabla^{A*}\psi^* + (mc/\hbar)^2\psi\psi^*. \tag{2.2}$$

Subtracting the free Klein–Gordon Hamiltonian,

$$H_{\psi}^{\mathrm{free}} = c^2\pi\pi^* - \mathrm{i}\hbar^{-1}V\psi\pi + \mathrm{i}\hbar^{-1}V\psi^*\pi^* + \nabla\psi\nabla\psi^* + (mc/\hbar)^2\psi\psi^*, \tag{2.3}$$

we find the interaction Hamiltonian expanded in first order in the tachyon charge,

$$\begin{aligned} H_{\psi}^{\mathrm{int}} := H_{\psi} - H_{\psi}^{\mathrm{free}} &= -\mathrm{i}\tilde{q}c^{-2}A_0\psi^*(\psi_{,t} + \mathrm{i}\hbar^{-1}V\psi) + \mathrm{i}\tilde{q}c^{-2}A_0\psi(\psi^*_{,t} - \mathrm{i}\hbar^{-1}V\psi^*) \\ &\quad + \mathrm{i}\tilde{q}\mathbf{A}\psi^*\nabla\psi - \mathrm{i}\tilde{q}\mathbf{A}\psi\nabla\psi^* + \mathrm{O}(\tilde{q}^2). \end{aligned} \tag{2.4}$$

The energy density of the free matter field follows from H_{ψ}^{free},

$$\rho_{\psi}^{\mathrm{free}} := c^{-2}\psi_{,t}\psi^*_{,t} - c^{-2}\hbar^{-2}V^2\psi\psi^* + \nabla\psi\nabla\psi^* + (mc/\hbar)^2\psi\psi^*. \tag{2.5}$$

We use $\nabla\psi\nabla\psi^* = -\psi^*\Delta\psi$, partially integrated, as well as the free field equation,

$$c^{-2}(\partial/\partial t + \mathrm{i}\hbar^{-1}V)^2\psi - \Delta\psi + (mc/\hbar)^2\psi = 0, \tag{2.6}$$

to write the energy density as

$$c^2\rho_{\psi}^{\mathrm{free}} = \psi_{,t}\psi^*_{,t} - \psi^*\psi_{,tt} - 2\mathrm{i}\hbar^{-1}V\psi^*\psi_{,t}. \tag{2.7}$$

The non-relativistic limit of (2.6), the Schrödinger equation in a Coulomb potential, is recovered by the substitution $\psi \to \hbar(2m)^{-1/2}\psi\exp(-\mathrm{i}mc^2t/\hbar)$ in the Lagrangian (2.1). The non-relativistic interaction is found by the same substitution in (2.4), cf [15].

We define the scalar product,

$$\langle\psi,\varphi\rangle := \mathrm{i}\int(\varphi^*\partial_t^V\psi - \psi\partial_t^{V*}\varphi^*)\,\mathrm{d}^3x, \qquad \partial_t^V := \partial_t + \mathrm{i}\hbar^{-1}V, \tag{2.8}$$

and note the continuity equation, $\rho_{,t} + \mathrm{div}\,\mathbf{j} = 0$, with the 4-current

$$\rho(\psi,\varphi) := \mathrm{i}q(\varphi^*\partial_t^V\psi - \psi\partial_t^{V*}\varphi^*), \qquad \mathbf{j}(\psi,\varphi) := -\mathrm{i}qc^2(\varphi^*\nabla\psi - \psi\nabla\varphi^*), \tag{2.9}$$

where ψ and φ are arbitrary wave fields solving the wave equation (2.6). We insert the separation ansatz $\psi_i = u_i \exp(-i\omega_i t)$ into the field equation (2.6) and the current (2.9), and define the shortcuts $\rho(\psi_m, \psi_n) =: \rho_{mn} \exp(-i\omega_{mn} t)$ and $\mathbf{j}(\psi_m, \psi_n) =: \mathbf{j}_{mn} \exp(-i\omega_{mn} t)$, where $\omega_{mn} := \omega_m - \omega_n$. In this way, we find the time separated wave equation

$$\Delta u_i = ((mc/\hbar)^2 - c^{-2}(\omega_i - \hbar^{-1} V)^2) u_i, \tag{2.10}$$

the matrix elements

$$\rho_{mn} = q(\omega_m + \omega_n - 2\hbar^{-1} V) u_m u_n^*, \qquad \mathbf{j}_{mn} = -iqc^2 (u_n^* \nabla u_m - u_m \nabla u_n^*), \tag{2.11}$$

and the orthonormality relation $\int \rho_{mn}\, d^3x = q\delta_{mn}$, where we use a normalization convenient for second quantization. The foregoing derivations apply to more or less any time independent potential V, no need for the spherical symmetry, just that the spectrum of (2.10) stays bounded from below. We do not elaborate on the continuous spectrum here, assuming box quantization. Scattering states are studied in section 3, more or less on the same footing.

We consider a free wave field, $\psi = \sqrt{\hbar} c \sum_n b_n u_n \, e^{-i\omega_n t}$, with arbitrary complex amplitudes b_n and positive frequencies ω_n, substitute it into the energy density $\rho_\psi^{\mathrm{free}}$ in (2.7), and find the field energy as $E = \int \rho_\psi^{\mathrm{free}}\, d^3x = \sum_n \hbar\omega_n b_n b_n^*$, where we made use of the orthonormality condition stated after (2.11). We restrict ourselves to positive frequency solutions, antiparticles can be dealt with analogously, there are no tachyonic antiparticles by the way, the superluminal Proca field is real. In (2.9), we put $\varphi = \psi$ and expand density and current

$$\rho(\psi) = \hbar c^2 \sum_{m,n} b_m b_n^* \rho_{mn} \, e^{-i\omega_{mn} t}, \qquad \mathbf{j}(\psi) = \hbar c^2 \sum_{m,n} b_m b_n^* \mathbf{j}_{mn} \, e^{-i\omega_{mn} t}, \tag{2.12}$$

with ρ_{mn} and $\mathbf{j}_{mn}$ as in (2.11). We substitute these series into the interaction Hamiltonian (2.4),

$$H_\psi^{\mathrm{int}} = -\frac{1}{\hbar c^3} (A_0 \rho(\psi) + \mathbf{A}\mathbf{j}(\psi)), \tag{2.13}$$

together with the Fourier series of the tachyon field,

$$\mathbf{A}(\mathbf{x}, t) = L^{-3/2} \sum_{\mathbf{k}} (\hat{\mathbf{A}}(\mathbf{k}) \exp(i(\mathbf{kx} - \omega t)) + \text{c.c.}), \qquad \hat{\mathbf{A}}(\mathbf{k}) := \sum_{\lambda=1}^{3} \varepsilon_{\mathbf{k},\lambda} \hat{a}(\mathbf{k}, \lambda), \tag{2.14}$$

where $\mathbf{k} := 2\pi\mathbf{n}/L$. The summation is over integer lattice points $\mathbf{n}$ in R^3, corresponding to periodic boundary conditions. The $\varepsilon_{\mathbf{k},1}$ and $\varepsilon_{\mathbf{k},2}$ are arbitrary real unit vectors (linear polarization vectors) orthogonal to $\varepsilon_{\mathbf{k},3} := \mathbf{k}_0 = \mathbf{k}/|\mathbf{k}|$, so that the $\varepsilon_{\mathbf{k},\lambda}$ constitute an orthonormal triad, and the $\hat{a}(\mathbf{k}, \lambda)$ are arbitrary complex numbers. The amplitudes $\hat{\mathbf{A}}$ can be arbitrarily prescribed, the time component A_0 of the potential is then determined by the Lorentz condition and the free field equations, and so is the dispersion relation, $k^2 = \omega^2/c^2 + m_t^2$. We split the potential into transversal and longitudinal components,

$$\hat{\mathbf{A}}^{\mathrm{T}}(\mathbf{k}) := \sum_{\lambda=1,2} \varepsilon_{\mathbf{k},\lambda} \hat{a}(\mathbf{k}, \lambda), \qquad \hat{A}_0^{\mathrm{T}} = 0, \tag{2.15}$$

$$\hat{\mathbf{A}}^{\mathrm{L}}(\mathbf{k}) := \mathbf{k}_0 \hat{a}(\mathbf{k}, 3), \qquad \hat{A}_0^{\mathrm{L}}(\mathbf{k}) = -c^2 k \omega^{-1} \hat{a}(\mathbf{k}, 3); \tag{2.16}$$

this decomposition is unique, as there is no gauge freedom. It is understood that $\omega(k)$ solves the mentioned dispersion relation. The time averaged energy densities of the wave fields (2.15) and (2.16) read

$$\langle \rho_{\mathrm{E}}^{\mathrm{T}} \rangle = 2c^{-2} \sum_{\mathbf{k};\lambda=1,2} \omega^2 \hat{a}\hat{a}^*, \qquad \langle \rho_{\mathrm{E}}^{\mathrm{L}} \rangle = -2m_t^2 \sum_{\mathbf{k}} \hat{a}(3)\hat{a}^*(3). \tag{2.17}$$

The averaging is essential to cancel the indefinite terms in the classical densities, mixtures of transversal and longitudinal modes. It is the time averaged energy densities rather than the classical Hamiltonian, that are quantized [15]. To this end, we introduce rescaled Fourier coefficients,

$$\hat{a}(\mathbf{k},\lambda) =: 2^{-1/2} c\hbar^{1/2}\omega^{-1/2} a_{\mathbf{k},\lambda}, \qquad \hat{a}(\mathbf{k},3) =: 2^{-1/2}\hbar^{1/2}\omega^{1/2} m_{\mathrm{t}}^{-1} a_{\mathbf{k},3}, \tag{2.18}$$

$\lambda = 1, 2$, so that the field energy gets a familiar shape

$$\langle \rho_{\mathrm{E}}^{\mathrm{T}} \rangle = \sum_{\mathbf{k};\lambda=1,2} \hbar\omega_k a_{\mathbf{k},\lambda} a^*_{\mathbf{k},\lambda}, \qquad \langle \rho_{\mathrm{E}}^{\mathrm{L}} \rangle = -\sum_{\mathbf{k}} \hbar\omega_k a_{\mathbf{k},3} a^*_{\mathbf{k},3}. \tag{2.19}$$

The Fourier coefficients $a_{\mathbf{k},\lambda}$ are interpreted as operators, and the complex conjugates $a^*_{\mathbf{k},\lambda}$ as their adjoints $a^+_{\mathbf{k},\lambda}$. We use commutation relations for the transversal degrees, $\lambda = 1, 2$, and anticommutators for the longitudinal modes, to turn the longitudinal energy in (2.19) into a positive definite operator. The fact that Fermi statistics is invoked to quantize a spin-one field in 4D seems strange at first sight, but the spin-statistics theorem is not applicable outside the light cone [3]. The occupation number representations of the energy densities can be found in [16], the vacuum is defined with regard to the universal rest frame, the comoving galaxy frame. The absolute spacetime defined by this reference frame is already required at the classical level [17], as the causality of the superluminal signal transfer is subject to the universal cosmic time order [12]. The galaxy frame is the rest frame of the cosmic absorber medium, the cosmic ether, a prerequisite for retarded wave propagation outside the light cone [13, 14]. The transversal Hamilton operator of the free tachyon field is $\langle \rho_{\mathrm{E}}^{\mathrm{T}} \rangle$ in (2.19), with the Fourier amplitudes $a_{\mathbf{k},\lambda} a^*_{\mathbf{k},\lambda}$ replaced by the operator products $a^+_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}$, the transversal modes being bosonic as in the zero-mass limit. The longitudinal energy operator is defined by the fermionic substitution $a_{\mathbf{k},3} a^*_{\mathbf{k},3} \to -a^+_{\mathbf{k},3} a_{\mathbf{k},3}$ in $\langle \rho_{\mathrm{E}}^{\mathrm{L}} \rangle$. To obtain the interaction operator, cf (2.13), we replace the amplitudes $b_m b_n^*$ of the Klein–Gordon field in (2.12) and (2.13) by bosonic operator products $b_n^+ b_m$. Similarly, the tachyonic field amplitudes $a^{(*)}_{\mathbf{k},\lambda}$ are replaced by bosonic or fermionic operators $a^{(+)}_{\mathbf{k},\lambda}$, as in the energy densities (2.19).

To sum up, there are three major deviations from the standard quantization procedure. Two of them are technical, the third is not, implying an absolute, non-relativistic spacetime conception to cope with superluminal quanta. First, the time averaged energy density rather than the classical Hamiltonian is taken as the starting point for the operator interpretation. Second, Fermi statistics is employed for the longitudinal modes of an integer spin field in 4D, and third, the vacuum state is defined with respect to the universal frame of reference, the absolute cosmic spacetime as manifested in the cosmic background radiations, the expanding galaxy grid, and the absorber medium.

We turn to the actual calculation of the transition rates, beginning with the transversal fields $\mathbf{A}^{\mathrm{T}}$, $A_0 = 0$, cf (2.14) and (2.15), in a fixed linear polarization λ (that is, no summation over λ in (2.15)). The transversal component of the interaction Hamiltonian (2.13) reads $H^{\mathrm{T}}_{\mathrm{int}} := -\hbar^{-1} c^{-3} \mathbf{A}^{\mathrm{T}} \mathbf{j}(\psi)$. We substitute the Fourier decompositions (2.14), (2.15), (2.18) and (2.12) into $\int H^{\mathrm{T}}_{\mathrm{int}}\, \mathrm{d}^3x$, and replace the amplitudes by operators $b_i^{(+)}$and $a^{(+)}_{\mathbf{k},\lambda}$. In this way, the T-matrix elements for absorption and emission can readily be identified as

$$\langle T^{\mathrm{T}}_{\mathrm{abs/em}} \rangle = -\frac{\hbar^{1/2}}{\sqrt{2}\omega_k^{1/2} L^{3/2}} \int \varepsilon_{\mathbf{k},\lambda} \mathbf{j}_{mn}\, \mathrm{e}^{\pm \mathrm{i}\mathbf{k}\mathbf{x}}\, \mathrm{d}^3x. \tag{2.20}$$

This corresponds to the absorption or emission of a transversal tachyon, cf [15] for details, where the interaction with a non-relativistic particle in a Coulomb potential was studied. The transition matrix for the interaction of tachyons with a free Klein–Gordon particle has been derived in [16]; we outline the changes necessary to incorporate the Coulomb potential and state the transition probabilities, mostly without derivation. The matrix elements $\langle T^{\mathrm{T}}_{\mathrm{abs,em}}\rangle$ in (2.20) only differ by a sign change in the exponential; the upper sign always refers to absorption. The tachyonic wave vector **k** relates to the tachyonic frequency ω_k by the dispersion relation stated before (2.15); the ω_k are positive, and the $\omega_{mn} := \omega_m - \omega_n$ refer to energy levels of the wave equation, cf (2.10). The initial state is denoted by a subscript m and the final state by n, so that a positive ω_{mn} stands for emission.

The longitudinal component of the interaction (2.13) reads $H^{\mathrm{L}}_{\mathrm{int}} = H^{\mathrm{L}(1)}_{\mathrm{int}} + H^{\mathrm{L}(2)}_{\mathrm{int}}$, where $H^{\mathrm{L}(1)}_{\mathrm{int}} = -\hbar^{-1}c^{-3}\mathbf{A}^{\mathrm{L}}\mathbf{j}(\psi)$ and $H^{\mathrm{L}(2)}_{\mathrm{int}} = -\hbar^{-1}c^{-3}A_0\rho(\psi)$, with the Fourier series of $\mathbf{A}^{\mathrm{L}}$ and A_0 defined in (2.14), (2.16) and (2.18). By substituting these series into $\int H^{\mathrm{L}(1)}_{\mathrm{int}}\,\mathrm{d}^3x$ and $\int H^{\mathrm{L}(2)}_{\mathrm{int}}\,\mathrm{d}^3x$, we find the respective T-matrix components,

$$\begin{aligned}\langle T^{\mathrm{L}(1)}_{\mathrm{abs/em}}\rangle &= -\frac{\hbar^{3/2}\omega_k^{1/2}}{\sqrt{2}m_{\mathrm{t}}c^2L^{3/2}}\int \mathbf{k}_0\mathbf{j}_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x,\\ \langle T^{\mathrm{L}(2)}_{\mathrm{abs/em}}\rangle &= \frac{\hbar^{3/2}k}{\sqrt{2}m_{\mathrm{t}}\omega_k^{1/2}L^{3/2}}\int \rho_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x.\end{aligned} \tag{2.21}$$

We have here restored the mass unit, $m_{\mathrm{t}} \to m_{\mathrm{t}}c/\hbar$, and $\mathbf{k}_0 = \mathbf{k}/k$ is the tachyonic unit wave vector. The longitudinal T-matrix, $\langle T^{\mathrm{L}}_{\mathrm{abs/em}}\rangle = \langle T^{\mathrm{L}(1)}_{\mathrm{abs/em}}\rangle + \langle T^{\mathrm{L}(2)}_{\mathrm{abs/em}}\rangle$, can be further simplified. To this end, we return to (2.8)–(2.11), and inspect the integral $\int (u_m\Delta u_n^* - u_n^*\Delta u_m)\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x$, once by applying the Gauss theorem, and once by employing the field equation (2.10). By making use of the continuity equation as stated after (2.8), we readily derive $\mathbf{k}_0\mathbf{j}_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}} = \mp k^{-1}\omega_{mn}\rho_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}$, valid under the integral sign, that is, up to a divergence. In this way, we can express the longitudinal T-matrix by the charge density only,

$$\langle T^{\mathrm{L}}_{\mathrm{abs/em}}\rangle = \frac{m_{\mathrm{t}}c^2}{\sqrt{2}\hbar^{1/2}\omega_k^{1/2}kL^{3/2}}\int \rho_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x, \tag{2.22}$$

where we used energy conservation, $\omega_k = \mp\omega_{mn}$, as well as the tachyonic dispersion relation $k^2 = \omega^2/c^2 + (m_{\mathrm{t}}c/\hbar)^2$.

We return to the transversal transition matrix (2.20). Once this matrix is known, the transition rate for the transversally induced absorption and emission in a given polarization λ is obtained by a standard procedure [18],

$$\mathrm{d}w^{\mathrm{T,ind}}_{\mathrm{abs/em}} \sim \frac{1}{8\pi^2}\frac{k}{\hbar c^2}\frac{1}{\mathrm{e}^{\beta\hbar\omega}-1}\left|\int \varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x\right|^2\mathrm{d}\Omega, \tag{2.23}$$

where ω (as well as $k(\omega)$) is taken at $|\omega_{mn}|$. The upper sign refers to absorption, and m to the initial state; the solid angle element, $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, is centred at the tachyonic wave vector **k**. The emission rate (2.23) also applies to spontaneous radiation, if the factor $(\mathrm{e}^{\beta\hbar\omega_k}-1)^{-1}$ (the averaged occupation number) is dropped, $\mathrm{d}w^{\mathrm{T,sp}}_{\mathrm{em}} \sim (\mathrm{e}^{\beta\hbar\omega}-1)\,\mathrm{d}w^{\mathrm{T,ind}}_{\mathrm{em}}$. The total emission rate is $\mathrm{d}w^{\mathrm{T}}_{\mathrm{em}} = \mathrm{d}w^{\mathrm{T,ind}}_{\mathrm{em}} + \mathrm{d}w^{\mathrm{T,sp}}_{\mathrm{em}}$. The spontaneous transversal emission rate is temperature independent, unaffected by the tachyonic heat bath, in contrast to the longitudinal emission discussed below. The rates for unpolarized radiation are obtained by replacing $\varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}$ in (2.23) by the transversal current, cf (2.11),

$$\mathbf{j}^{\mathrm{T}}_{mn} := -\mathrm{i}qc^2(u_n^*\nabla^{\mathrm{T}}u_m - u_m\nabla^{\mathrm{T}}u_n^*), \qquad \nabla^{\mathrm{T}} := \nabla - \mathbf{k}_0(\mathbf{k}_0\cdot\nabla), \tag{2.24}$$

with $\mathbf{k}_0 := \mathbf{k}/k$. The spontaneous emission rate is time symmetric, applying to tachyon absorption as well.

The longitudinal transition rate is assembled with $\langle T^{\mathrm{L}}_{\mathrm{abs/em}}\rangle$ in (2.22),

$$\mathrm{d}w^{\mathrm{L,ind}}_{\mathrm{abs/em}} \sim \frac{1}{8\pi^2}\frac{m_{\mathrm{t}}^2c^2}{\hbar^3k}\frac{1}{\mathrm{e}^{\beta\hbar\omega}+1}\left|\int \rho_{mn}\,\mathrm{e}^{\pm\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x\right|^2\mathrm{d}\Omega. \tag{2.25}$$

Here, m denotes the initial state, both for absorption and emission, and $\omega = |\omega_{mn}|$. The total emission rate reads as $\mathrm{d}w_{\mathrm{em}}^{\mathrm{L}} = \mathrm{d}w_{\mathrm{em},T=0}^{\mathrm{L,sp}} - \mathrm{d}w_{\mathrm{em}}^{\mathrm{L,ind}}$, with $\mathrm{d}w_{\mathrm{em}}^{\mathrm{L,ind}}$ in (2.25) and $\mathrm{d}w_{\mathrm{em},T=0}^{\mathrm{L,sp}} := (\mathrm{e}^{\beta\hbar\omega} + 1)\,\mathrm{d}w_{\mathrm{em}}^{\mathrm{L,ind}}$, the latter being the spontaneous transition rate in the zero temperature limit [15]. At finite temperature, the spontaneous emission rate is $\mathrm{d}w_{\mathrm{em}}^{\mathrm{L,sp}} = \mathrm{d}w_{\mathrm{em},T=0}^{\mathrm{L,sp}} - 2\mathrm{d}w_{\mathrm{em}}^{\mathrm{L,ind}}$, so that the total emission reads $\mathrm{d}w_{\mathrm{em}}^{\mathrm{L}} = \mathrm{d}w_{\mathrm{em}}^{\mathrm{L,ind}} + \mathrm{d}w_{\mathrm{em}}^{\mathrm{L,sp}}$. Hence,

$$\mathrm{d}w_{\mathrm{em}}^{\mathrm{L,sp}} \sim (\mathrm{e}^{\beta\hbar\omega} - 1)\,\mathrm{d}w_{\mathrm{em}}^{\mathrm{L,ind}} = \tanh(\beta\hbar\omega/2)\,\mathrm{d}w_{\mathrm{em},T=0}^{\mathrm{L,sp}}. \tag{2.26}$$

The time symmetry of the transition rates also extends to the longitudinal radiation; they stay invariant with regard to an interchange of the indices m and n (representing initial and final states) accompanied by a sign change of the wave vector $\mathbf{k}$. The longitudinal emission (2.26) is temperature dependent and vanishes in the high-temperature limit.

The dipole approximation of the tachyonic transition rates (2.23) and (2.25) allows a quantitative comparison with electromagnetic radiation, avoiding the explicit evaluation of matrix elements; otherwise this has to be done on a case by case basis, for each transition. We start with $\int (u_m \Delta u_n^* - u_n^* \Delta u_m)\mathbf{x}\,\mathrm{d}^3x$, apply the Gauss theorem once and the time separated field equation (2.10) once, and find, cf (2.24),

$$\int \mathbf{j}_{mn}^{(\mathrm{T})}\,\mathrm{d}^3x = -\mathrm{i}\omega_{mn}\mathbf{d}_{mn}^{(\mathrm{T})}, \qquad \mathbf{d}_{mn} := \int \rho_{mn}\mathbf{x}\,\mathrm{d}^3x, \qquad \mathbf{d}_{mn}^{\mathrm{T}} := \mathbf{d}_{mn} - \mathbf{k}_0(\mathbf{k}_0\mathbf{d}_{mn}). \tag{2.27}$$

We consider unpolarized transversal radiation, replacing in the transition rate (2.23) $\varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}$ by the transversal current $\mathbf{j}_{mn}^{\mathrm{T}}$. In (2.23) we drop the exponential, and in the longitudinal rate (2.25) we expand it in first order and substitute the dipole moment (2.27), so that the angular integrations can readily be carried out by making use of $\int \sin^2\theta\,\mathrm{d}\Omega = 8\pi/3$. Hence,

$$\int \left| \int \mathbf{j}_{mn}^{\mathrm{T}}\,\mathrm{d}^3x \right|^2 \mathrm{d}\Omega = \frac{8\pi}{3}\omega_{mn}^2|\mathbf{d}_{mn}|^2, \qquad \int \left| \int \rho_{mn}\mathbf{k}\mathbf{x}\,\mathrm{d}^3x \right|^2 \mathrm{d}\Omega = \frac{4\pi}{3}k^2|\mathbf{d}_{mn}|^2, \tag{2.28}$$

from which the dipole approximation of the induced rates (2.23) and (2.25) follows,

$$w_{\mathrm{abs/em}}^{\mathrm{T,ind}} \sim \frac{4}{3}\frac{|\mathbf{d}_{mn}|^2}{4\pi}\frac{k\omega^2}{\hbar c^2}\frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1}, \qquad w_{\mathrm{abs/em}}^{\mathrm{L,ind}} \sim \frac{2}{3}\frac{|\mathbf{d}_{mn}|^2}{4\pi}\frac{m_{\mathrm{t}}^2c^2k}{\hbar^3}\frac{1}{\mathrm{e}^{\beta\hbar\omega} + 1}. \tag{2.29}$$

The spontaneous dipole emission rates read $w_{\mathrm{em}}^{\mathrm{T,sp}} \sim (\mathrm{e}^{\beta\hbar\omega} - 1)w_{\mathrm{em}}^{\mathrm{T,ind}}$, cf after (2.23), and $w_{\mathrm{em}}^{\mathrm{L,sp}} \sim (\mathrm{e}^{\beta\hbar\omega} - 1)w_{\mathrm{em}}^{\mathrm{L,ind}}$, cf (2.26). We thus find the ratios,

$$\frac{w_{\mathrm{abs}}^{\mathrm{L}}}{w_{\mathrm{abs}}^{\mathrm{T}}} \sim \frac{w_{\mathrm{em}}^{\mathrm{L}}}{w_{\mathrm{em}}^{\mathrm{T}}} \sim \frac{1}{2}\tanh\left(\frac{\beta\hbar\omega}{2}\right)\frac{m_{\mathrm{t}}^2c^4}{\hbar^2\omega^2}, \tag{2.30}$$

for induced as well as spontaneous radiation. The dipole rates (2.29) refer to a transition frequency $\omega = |\omega_{mn}|$, they are angular integrated, and the transversal radiation is unpolarized. The electromagnetic transition rates $w_{\mathrm{abs/em}}^{\mathrm{ph,ind}}$ are recovered from the transversal rates by dropping the tachyon mass in the wave vector and replacing the tachyonic charge q in the dipole moment by its electric counterpart. Hence,

$$\frac{w^{\mathrm{T}}}{w^{\mathrm{ph}}} \sim \frac{q^2}{e^2}\frac{\sqrt{\hbar^2\omega^2 + m_{\mathrm{t}}^2c^4}}{\hbar\omega} \tag{2.31}$$

is valid for induced and spontaneous radiation alike. The following estimates are based on the dipole approximations (2.30) and (2.31), the tachyon mass $m_{\mathrm{t}} \approx 2.15$ keV$/c^2$, and the ratio $q^2/e^2 \approx 1.4 \times 10^{-11}$ of tachyonic and electric fine structure constants. (The latter are upper bounds inferred from Lamb shifts in hydrogen and hydrogen-like ions [10].) The peak of the longitudinal spectral density of a free tachyon gas is located at $\beta h\nu_{\max} \approx 2.218$; this frequency also lies, for any temperature, in the bulk of the photon and transversal tachyon distributions.

Assuming that equilibrium has been reached, we may identify the Ly-α lines of hydrogen (10.2 eV) with the spectral peak $\nu_{\max}$, corresponding to a temperature of $kT(\nu_{\max}) \approx 4.6$ eV. We thus find $w^{\mathrm{L}}/w^{\mathrm{T}}(\nu_{\max}) \approx 1.8 \times 10^4$ and $w^{\mathrm{T}}/w^{\mathrm{ph}}(\nu_{\max}) \approx 3.0 \times 10^{-9}$. For comparison, the Ly-α_1 transition in hydrogenic uranium ($\nu_{\max} = 0.23$ MeV) results in a temperature of $kT(\nu_{\max}) \approx 0.1$ MeV, so that $w^{\mathrm{L}}/w^{\mathrm{T}}(\nu_{\max}) \approx 3.6 \times 10^{-5}$ and $w^{\mathrm{T}}/w^{\mathrm{ph}}(\nu_{\max}) \approx 1.4 \times 10^{-11}$. Clearly, something has to be done to overcome the tiny ratio of the fine structure constants. In section 4 we invoke Rydberg states to that effect.

3. Transversal and longitudinal ionization cross sections

We study tachyonic photoelectric effect, the ejection of a bound electron into the continuum by the tachyon absorption. We start with the transition rates [15],

$$w_{\mathrm{abs}}^{\mathrm{T,L}} \underset{t\to\infty}{\sim} \frac{n_{\mathbf{k}}}{t\hbar^2} \sum_{\mathbf{k}_n} \left|\langle T_{\mathrm{abs}}^{\mathrm{T,L}}\rangle\right|^2 \left| \int_{-t/2}^{t/2} \mathrm{e}^{-\mathrm{i}(\omega_{mn}+\omega_k)t}\, \mathrm{d}t \right|^2$$

$$\sim \frac{2\pi n_{\mathbf{k}}}{\hbar^2 c^2} \frac{L^3}{(2\pi)^3} \int \mathrm{d}\Omega \int_{mc^2/\hbar}^{\infty} \left|\langle T_{\mathrm{abs}}^{\mathrm{T,L}}\rangle\right|^2 \delta(\omega_m - \omega_n + \omega_k) k_n \omega_n \,\mathrm{d}\omega_n, \tag{3.1}$$

where we replaced the summation over the electronic wave vectors by the continuum limit, $L^3(2\pi)^{-3}\int \mathrm{d}\mathbf{k}_n$, and used the subluminal dispersion relation, $k_n^2 = \omega_n^2/c^2 - (mc/\hbar)^2$, to obtain $\mathrm{d}\mathbf{k}_n = c^{-2}k_n\omega_n\,\mathrm{d}\omega_n\,\mathrm{d}\Omega$. The occupation numbers $n_{\mathbf{k}}$ refer to the incident tachyon flux, and the absorption rates are readily assembled by means of the transition matrices (2.20) and (2.22),

$$\mathrm{d}w_{\mathrm{abs}}^{\mathrm{T}} \sim \frac{n_{\mathbf{k}}}{8\pi^2} \frac{1}{\hbar c^2} \frac{k_n\omega_n}{\omega_k} \,\mathrm{d}\Omega \left| \int \varepsilon_{\mathbf{k},\lambda} \mathbf{j}_{mn}\, \mathrm{e}^{\mathrm{i}\mathbf{kx}}\, \mathrm{d}^3x \right|^2, \tag{3.2}$$

$$\mathrm{d}w_{\mathrm{abs}}^{\mathrm{L}} \sim \frac{n_{\mathbf{k}}}{8\pi^2} \frac{m_{\mathrm{t}}^2 c^2}{\hbar^3} \frac{k_n\omega_n}{k^2\omega_k} \,\mathrm{d}\Omega \left| \int \rho_{mn}\, \mathrm{e}^{\mathrm{i}\mathbf{kx}}\, \mathrm{d}^3x \right|^2, \tag{3.3}$$

with $\omega_n = \omega_m + \omega_k$ substituted. The current density of the tachyon flux is $\upsilon_{\mathrm{gr}} n_{\mathbf{k}}/L^3$, with $\upsilon_{\mathrm{gr}} = c^2k/\omega_k$, so that the cross sections relate to the absorption rates as

$$\mathrm{d}\sigma^{\mathrm{T,L}} = \frac{L^3\omega_k}{c^2 k n_{\mathbf{k}}} \,\mathrm{d}w_{\mathrm{abs}}^{\mathrm{T,L}}. \tag{3.4}$$

In section 2 and in (3.1)–(3.3), we have denoted the electronic parameters of the final state, the ejected electron, by a subscript n, such as k_n and ω_n. In the following explicit calculations, we use a subscript e instead, so that u_{e} stands for the wavefunction of the final electronic state. (When studying Rydberg states in section 4, we use the subscript n to label the principal quantum number of the initial bound state, therefore this change of notation.) The initial electronic bound state is indicated, throughout this section, by a subscript m or a subscript zero, if we consider the ground state. The quantities $\mathbf{k}$ and ω_k always refer to the ionizing or, in recombination, emitted tachyon. We content ourselves with the non-relativistic version of the cross sections (3.4). In fact, the most interesting interaction takes place at the ionization threshold, where the ejected electron has nearly zero energy, cf section 4. Here, we discuss ground state ionization in Born and dipole approximations.

We start with the Born (plane wave) approximation. In this limit, the cross sections are fairly easy to derive, and we study their scaling with the tachyonic and electric fine structure constants, as well as their dependence on the tachyon mass. The current in (3.2) reads $\mathbf{j}_{mn} = -\mathrm{i}qc^2(u_e^*\nabla u_m - u_m\nabla u_e^*)$, cf (2.11). The initial bound state u_m is taken as a solution of the Schrödinger equation in a Coulomb potential, normalized to $\int |u_m|^2\, \mathrm{d}^3x \approx \hbar/(2mc^2)$, cf after (2.11). We approximate $\hbar\omega_m \approx mc^2 - E_m$, where E_m is the non-relativistic bound state energy, defined positive. The ionization energy of the ground state is $E_0 = mc^2\alpha_Z^2/2$, where $\alpha_Z \approx Z/137$. The final state in the continuum is approximated by a plane wave, $u_{\rm e} \approx (2\omega_{\rm e})^{-1/2}L^{-3/2}\,{\rm e}^{{\rm i}\mathbf{k}_{\rm e}\mathbf{x}}$, which amounts to dropping the Coulomb potential in the wave equation (2.10) and to using box-integration when normalizing. The criterion for the applicability of the Born approximation in a Coulomb potential reads $\alpha_Z = Ze^2/(4\pi\hbar c) \ll \upsilon_{\rm e}/c$, where $\upsilon_{\rm e}$ is the speed of the ejected electron, $\upsilon_{\rm e}/c \ll 1$. Thus, $E_0 \ll m\upsilon_{\rm e}^2/2$, and we may approximate $\hbar\omega_{\rm e} \approx mc^2 + m\upsilon_{\rm e}^2/2$ and $\hbar k_{\rm e} \approx m\upsilon_{\rm e}$. This is equivalent to $\frac{1}{2}|V(r_{\rm B})| = E_0 \ll \hbar\omega_k$, with $r_{\rm B} := \hbar/(mc\alpha_Z)$ and V as in (2.1), since energy conservation requires $\omega_{\rm e} = \omega_m + \omega_k$, which allows us to approximate $\hbar\omega_k \approx m\upsilon_{\rm e}^2/2$. This condition, $E_0 \ll \hbar\omega_k$, defines a lower bound on the tachyonic frequency for the Born approximation to be valid, in addition to $\hbar\omega_k \ll mc^2$.

To obtain the transversal cross section, we substitute

$$\varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}\,{\rm e}^{{\rm i}\mathbf{kx}} \approx \sqrt{2}L^{-3/2}\omega_{\rm e}^{-1/2}qc^2\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\rm e}u_m\,{\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}} \tag{3.5}$$

into the absorption rate (3.2). This identity is valid, up to a divergence, for the transversal degrees $\lambda = 1, 2$, and we arrive at

$$d\sigma_\lambda^{\rm T} \sim \frac{1}{\pi}\frac{q^2}{4\pi}\frac{k_{\rm e}}{\hbar k}(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\rm e})^2\,d\Omega\left|\int u_m\,{\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}\,d^3x\right|^2, \tag{3.6}$$

which accounts for the ionization by linearly polarized transversal tachyons. To find the transversal polarization average, we replace $\varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}$ in (3.2) by the transversal current $\mathbf{j}_{mn}^{\rm T}$, cf (2.24), and multiply $dw_{\rm abs}^{\rm T}$ by a factor of $1/2$. Accordingly, the unpolarized transversal cross section is obtained by replacing $(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\rm e})^2$ in (3.6) by $\frac{1}{2}(\mathbf{k}_{\rm e} - \mathbf{k}_0(\mathbf{k}_0\mathbf{k}_{\rm e}))^2 \equiv \frac{1}{2}k_{\rm e}^2\sin^2\theta$, where the tachyonic unit wave vector, $\mathbf{k}_0 = \mathbf{k}/k$, defines the polar axis.

To proceed further, we have to specify the bound state u_m. We consider the simplest case, the ground state eigenfunction, $u_0 = \left(2\pi r_{\rm B}^3 mc^2/\hbar\right)^{-1/2}{\rm e}^{-r/r_{\rm B}}$. The integration in (3.6) is readily done,

$$\int u_0\,{\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}\,d^3x \approx \frac{8\pi}{r_{\rm B}}\frac{1}{\left(2\pi r_{\rm B}^3 mc^2/\hbar\right)^{1/2}}\frac{1}{k_{\rm e}^4}\left(1 + 4\frac{\mathbf{k}_{\rm e}\mathbf{k}}{k_{\rm e}^2}\right), \tag{3.7}$$

where we expanded in $k/k_{\rm e}$ as well as $1/(k_{\rm e}r_{\rm B})$. Hence,

$$d\sigma_\lambda^{\rm T} \sim 2^5\alpha_{\rm q}\hbar\frac{\sin^2\theta\cos^2\varphi}{mcr_{\rm B}^5k_{\rm e}^5k}\left(1 + 8\frac{k}{k_{\rm e}}\cos\theta\right)d\Omega, \tag{3.8}$$

where $\alpha_{\rm q} := q^2/(4\pi\hbar c) \approx 1.0 \times 10^{-13}$ is the tachyonic fine structure constant, cf after (2.31). As $\cos\theta = \mathbf{k}_0\mathbf{k}_{{\rm e},0}$, we can write $\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{{\rm e},0} = \sin\theta\cos\varphi$ in polar angles. The subscript zeros denote unit vectors, and λ labels the two linear polarizations. The Born approximation and the non-relativistic limit require $k_{\rm e}r_{\rm B} \approx \upsilon_{\rm e}/(c\alpha_Z) \gg 1$, cf after (3.4). The tachyonic dispersion relation gives $\hbar k \approx \varepsilon/c$, where $\varepsilon^2 = \left(m\upsilon_{\rm e}^2/2\right)^2 + m_{\rm t}^2c^4$, cf after (2.14). Since $m_{\rm t}/m \approx 1/238 < \alpha_Z$, we have $\upsilon_{\rm e}/c \gg m_{\rm t}/m$, and thus $k/k_{\rm e} \approx \varepsilon/(mc\upsilon_{\rm e}) \ll 1$, which justifies the expansions in (3.7). The two energies determining ε can be of comparable magnitude, as illustrated in the example below, after (3.15). The photonic cross section $d\sigma_\lambda^{\rm ph}$ is recovered from (3.8) by dropping the mass term in the tachyonic wave vector k and by replacing $\alpha_{\rm q} \to \alpha_{\rm e}$.

In the longitudinal absorption rate (3.3), we may approximate $\rho_{mn} \approx q(\omega_m + \omega_{\rm e})u_m u_e^*$, cf (2.11), to obtain the longitudinal cross section, cf (3.4),

$$d\sigma^{\rm L} \sim \frac{1}{\pi}\frac{q^2}{4\pi}\frac{m_{\rm t}^2m^2c^4}{\hbar^5}\frac{k_{\rm e}}{k^3}\,d\Omega\left|\int u_m\,{\rm e}^{{\rm i}\mathbf{kx}}\left({\rm e}^{-{\rm i}\mathbf{k}_{\rm e}\mathbf{x}} + \psi_{k_{\rm e}}^*\right)d^3x\right|^2. \tag{3.9}$$

The plane wave approximation is not quite sufficient for the longitudinal radiation, and we have indicated the first-order correction in α_Z, satisfying $\left(\Delta + k_{\mathrm{e}}^2\right)\psi_{k_{\mathrm{e}}} = -2\mathrm{e}^{-\mu r + \mathrm{i}\mathbf{k}_{\mathrm{e}}\mathbf{x}}/(r r_{\mathrm{B}})$. Here, an exponential convergence factor has temporarily been inserted into the Coulomb potential. We multiply this equation with $\mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{x}}$, integrate spatially, and make use of $\int r^{-1}\,\mathrm{e}^{-\mu r - \mathrm{i}\mathbf{q}\mathbf{x}}\,\mathrm{d}^3x = 4\pi(\mathbf{q}^2 + \mu^2)^{-1}$ to find, in momentum space,

$$\hat{\psi}_{k_{\mathrm{e}}}(\mathbf{q}) = \frac{8\pi}{r_{\mathrm{B}}}\frac{1}{(\mathbf{q} - \mathbf{k}_{\mathrm{e}})^2(\mathbf{q}^2 - \mathbf{k}_{\mathrm{e}}^2)}. \tag{3.10}$$

The convention for Fourier transforms is $\hat{\psi}_k(\mathbf{q}) := \int \psi_k(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\mathbf{q}\mathbf{x}}\,\mathrm{d}^3x$. The wavefunction, $(2\pi)^3\delta(\mathbf{q} - \mathbf{k}_{\mathrm{e}}) + \hat{\psi}_{k_{\mathrm{e}}}(\mathbf{q})$, is normalized to $(2\pi)^3\delta$, up to terms of $\mathrm{O}(\alpha_Z^2)$, like the plane wave. The Fourier transform of the ground state u_0, cf before (3.7), is readily obtained via $\int \mathrm{e}^{-\mu r - \mathrm{i}\mathbf{q}\mathbf{x}}\,\mathrm{d}^3x = 8\pi\mu(\mathbf{q}^2 + \mu^2)^{-2}$. This is apparently a limit definition, $(2\pi)^3\delta(\mathbf{q}, \mu \to 0)$, of the δ-function, which can be used to approximate the exact Fourier transform

$$\hat{u}_0(\mathbf{q}) = \frac{8\pi}{r_{\mathrm{B}}}\frac{\left(2\pi r_{\mathrm{B}}^3 mc^2/\hbar\right)^{-1/2}}{\left(\mathbf{q}^2 + r_{\mathrm{B}}^{-2}\right)^2}. \tag{3.11}$$

In this way, a cumbersome convolution can be greatly simplified in the momentum space representation of the matrix element

$$\int u_0\,\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}\left(\mathrm{e}^{-\mathrm{i}\mathbf{k}_{\mathrm{e}}\mathbf{x}} + \psi_{k_{\mathrm{e}}}^*\right)\mathrm{d}^3x \approx \left(2\pi r_{\mathrm{B}}^3 mc^2/\hbar\right)^{-1/2}\frac{16\pi}{r_{\mathrm{B}}}\frac{\mathbf{k}_{\mathrm{e}}\mathbf{k}}{k_{\mathrm{e}}^6}. \tag{3.12}$$

Here, we have again expanded in k/k_{e} and $1/(k_{\mathrm{e}}r_{\mathrm{B}})$. Substituting this into (3.9), we find the longitudinal cross section

$$\mathrm{d}\sigma^{\mathrm{L}} \sim 2^7\alpha_{\mathrm{q}}\frac{m_{\mathrm{t}}^2 mc^3}{r_{\mathrm{B}}^5\hbar^3 k k_{\mathrm{e}}^9}\cos^2\theta\,\mathrm{d}\Omega, \tag{3.13}$$

where $\cos\theta = \mathbf{k}_0\mathbf{k}_{\mathrm{e},0}$, as above.

We turn to the extrema of $\mathrm{d}\sigma^{\mathrm{T,L}}/\mathrm{d}\theta\,\mathrm{d}\varphi$. The maximum of the transversal section (3.8) (with the relativistic k/k_{e}-correction dropped) and the longitudinal minimum, cf (3.13), occur at $\theta_{\max}^{\mathrm{T}} \approx \theta_{\min}^{\mathrm{L}} \approx \pi/2$. The longitudinal maxima are the roots of $\sin\theta_{\max}^{\mathrm{L}} = 1/\sqrt{3}$, that is, $\theta_{\max}^{\mathrm{L}} \approx 0.6155$ and $\pi - \theta_{\max}^{\mathrm{L}}$. Thus the peaks of the longitudinal section (3.13) occur at scattering angles of 35.3° and 144.7° and have the same height. This clear separation of the transversal and longitudinal maxima can be used to determine the polarization of tachyon radiation, in particular to identify longitudinal quanta, even in a flux overwhelmingly photonic.

The angular integrations of the differential cross sections (3.8) and (3.13) can easily be carried out,

$$\sigma^{\mathrm{T}} \sim \frac{2^7\pi}{3}\frac{\alpha_{\mathrm{q}}\hbar^2}{m r_{\mathrm{B}}^5 k_{\mathrm{e}}^5\varepsilon}, \qquad \sigma^{\mathrm{L}} \sim \frac{2^9\pi}{3}\frac{\alpha_{\mathrm{q}} m_{\mathrm{t}}^2 mc^4}{\hbar^2 r_{\mathrm{B}}^5 k_{\mathrm{e}}^9\varepsilon}, \tag{3.14}$$

where $\varepsilon = \sqrt{\left(m\upsilon_{\mathrm{e}}^2/2\right)^2 + m_{\mathrm{t}}^2c^4}$. The electronic and tachyonic wave vectors relate to the speed of the ejected electron as $\hbar k_{\mathrm{e}} \approx m\upsilon_{\mathrm{e}}$ and $\hbar k \approx \varepsilon/c$, respectively, since the ionization energy is neglected in the Born approximation. The total photonic cross section σ^{ph} is recovered from the transversal section by dropping the tachyon mass in ε and identifying tachyonic with electric charge. The ratios of the total cross sections read

$$\frac{\sigma^{\mathrm{T}}}{\sigma^{\mathrm{ph}}} \sim \frac{q^2}{e^2}\frac{m\upsilon_{\mathrm{e}}^2}{2\varepsilon}, \qquad \frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{T}}} \sim 4\frac{m_{\mathrm{t}}^2 m^2 c^4}{\hbar^4 k_{\mathrm{e}}^4}, \tag{3.15}$$

where $q^2/e^2 \approx 1.4 \times 10^{-11}$ and $m_{\mathrm{t}}/m \approx 1/238$, cf after (2.31) and (3.8). For instance, if $\upsilon_{\mathrm{e}}/c \approx 0.1$, we find $\varepsilon \approx mc^2/153$, so that the transversal, longitudinal and photoelectric sections compare as $\sigma^{\mathrm{T}}/\sigma^{\mathrm{ph}} \approx 1.1 \times 10^{-11}$ and $\sigma^{\mathrm{L}}/\sigma^{\mathrm{T}} \approx 0.71$.

To reach the ionization threshold, we turn to the dipole approximation of the cross sections (3.2)–(3.4), using the exact eigenfunction of the scattering state. In the absorption rate (3.2), we substitute the current defined after (3.4) and in (2.11), and expand (i.e. drop) the exponential in the integral, replacing

$$\varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}} \approx -2\mathrm{i}qc^2u_{\mathrm{e}}^{*}\varepsilon_{\mathbf{k},\lambda}\nabla u_m. \tag{3.16}$$

We take the ground state eigenfunction u_0 defined before (3.7) as initial state u_m. The final state is the exact Coulomb wavefunction with the asymptotic limit $u_{\mathrm{e}} \sim (2\omega_{\mathrm{e}})^{-1/2}L^{-3/2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}_{\mathrm{e}}\mathbf{x}}$, which admits the expansion [19–23],

$$u_{\mathrm{e}} = \frac{1}{2(2\omega_{\mathrm{e}})^{1/2}L^{3/2}k_{\mathrm{e}}r_{\mathrm{B}}}\sum_{l=0}^{\infty} i^l(2l+1)\,\mathrm{e}^{-\mathrm{i}\delta_l}R_{k_{\mathrm{e}},l}(r)P_l(\cos\theta), \tag{3.17}$$

where $\cos\theta = \mathbf{k}_{\mathrm{e},0}\mathbf{r}_0$ and $k_{\mathrm{e}} = |\mathbf{k}_{\mathrm{e}}|$. The wavefunctions (3.17) are normalized to $(2\omega_{\mathrm{e}})^{-1}(2\pi/L)^3\delta(\mathbf{k}-\mathbf{k}')$. The phase shifts δ_l will not be needed in the following. We note $P_0 = 1$ and $P_1 = \cos\theta$, as well as the radial $l = 1$ function,

$$R_{k,1} = \frac{2\sqrt{2\pi}}{3r_{\mathrm{B}}^{1/2}}\sqrt{\frac{k\left(1+k^2r_{\mathrm{B}}^2\right)}{1-\mathrm{e}^{-2\pi/(kr_{\mathrm{B}})}}}\,r\,\mathrm{e}^{-\mathrm{i}kr}\,{}_1F_1\left(2+\frac{\mathrm{i}}{kr_{\mathrm{B}}};4;2\mathrm{i}\,kr\right). \tag{3.18}$$

The orthogonality relation,

$$\int P_i(\varepsilon_{\mathbf{k},\lambda}\mathbf{r}_0)P_l(\mathbf{k}_{\mathrm{e},0}\mathbf{r}_0)\,\mathrm{d}\Omega_{\mathbf{r}_0} = \frac{4\pi\,\delta_{il}}{2l+1}P_l(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0}), \tag{3.19}$$

(solid angle integration) leads to selection rules, so that only the $l = 1$ term in (3.17) is required. In fact, by means of (3.16), (3.18) and (3.19),

$$\int \varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}\,\mathrm{d}^3x \approx \frac{4\pi qc^2\,\mathrm{e}^{\mathrm{i}\delta_1}}{\sqrt{2\omega_{\mathrm{e}}}k_{\mathrm{e}}r_{\mathrm{B}}^2L^{3/2}}\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0}\int_0^{\infty}u_0R_{k_{\mathrm{e}},1}^{*}r^2\,\mathrm{d}r, \tag{3.20}$$

where $\omega_{\mathrm{e}} \approx mc^2/\hbar$, and u_0 is the ground state normalized as stated before (3.7). The integration over the confluent hypergeometric function is standard,

$$\int_0^{\infty} r^{3+n}\,\mathrm{d}r\,\mathrm{e}^{-(1/r_{\mathrm{B}}-\mathrm{i}k_{\mathrm{e}})r}\,{}_1F_1\left(2-\frac{\mathrm{i}}{k_{\mathrm{e}}r_{\mathrm{B}}};4;-2\mathrm{i}k_{\mathrm{e}}r\right) = \frac{\Gamma(4+n)r_{\mathrm{B}}^{4+n}}{\left(1+k_{\mathrm{e}}^2r_{\mathrm{B}}^2\right)^2}\left(\frac{1-\mathrm{i}k_{\mathrm{e}}r_{\mathrm{B}}}{1+\mathrm{i}k_{\mathrm{e}}r_{\mathrm{B}}}\right)^{-\mathrm{i}/(k_{\mathrm{e}}r_{\mathrm{B}})}$$

$$\times(1-\mathrm{i}k_{\mathrm{e}}r_{\mathrm{B}})^{-n}\,{}_2F_1\left(-n,2-\frac{\mathrm{i}}{k_{\mathrm{e}}r_{\mathrm{B}}};4;\frac{2\mathrm{i}k_{\mathrm{e}}r_{\mathrm{B}}}{1+\mathrm{i}k_{\mathrm{e}}r_{\mathrm{B}}}\right), \tag{3.21}$$

where ${}_2F_1$ on the right-hand side is a polynomial of order n. We put $n = 0$, and assemble the cross sections (3.4) via (3.2), (3.20), (3.18) and (3.21) as

$$\mathrm{d}\sigma_\lambda^{\mathrm{T}} \sim \frac{2^6\pi\alpha_{\mathrm{q}}\hbar b(k_{\mathrm{e}}r_{\mathrm{B}})}{mck\left(1+k_{\mathrm{e}}^2r_{\mathrm{B}}^2\right)^3}(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0})^2\,\mathrm{d}\Omega, \qquad b(x) := \frac{\exp(-4\arctan x/x)}{1-\exp(-2\pi/x)}. \tag{3.22}$$

Here, α_{q} is the tachyonic fine structure constant defined after (3.8), and the exponential containing the arctan is just the square of the factor with the imaginary exponent on the right-hand side of (3.21), $2\mathrm{i}\arctan x = \log((1+\mathrm{i}x)/(1-\mathrm{i}x))$. Principal values are implied, $0 \leqslant \arctan \leqslant \pi/2$, and $b(x)$ admits the expansions $\mathrm{e}^{-4}(1+4x^2/3+\cdots)$ and $(2\pi)^{-1}x(1-\pi/x+\cdots)$. The tachyonic energy $\hbar\omega$ accounts for the ionization energy E_0 and the kinetic energy $E_{\mathrm{e}} = \hbar^2k_{\mathrm{e}}^2\big/(2m)$ of the ejected electron, so that $\hbar\omega/E_0 = 1+k_{\mathrm{e}}^2r_{\mathrm{B}}^2$, where $E_0 = \hbar^2\big/\left(2mr_{\mathrm{B}}^2\right)$. The tachyonic dispersion relation reads $c\hbar k = \sqrt{(\hbar\omega)^2+(m_{\mathrm{t}}c^2)^2}$, cf after (2.14). Hence,

$$\mathrm{d}\sigma_\lambda^{\mathrm{T}} \sim \frac{2^7\pi\alpha_{\mathrm{q}}r_{\mathrm{B}}^2E_0^4b(k_{\mathrm{e}}r_{\mathrm{B}})}{(\hbar\omega)^3\sqrt{(\hbar\omega)^2+(m_{\mathrm{t}}c^2)^2}}(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0})^2\,\mathrm{d}\Omega. \tag{3.23}$$

The Born approximation (3.8) is recovered in the limit $k_e r_B \gg 1$, if we put $\hbar\omega/E_0 \approx k_e^2 r_B^2$ and expand $b(x)$; only the small relativistic correction $\propto k/k_e$ escapes in the dipole approximation.

We turn to the longitudinal cross section (3.4), expand the exponential in (3.3) to first order, and replace $\rho_{mn}\,\mathrm{e}^{\mathrm{i}\mathbf{kx}} \approx 2\mathrm{i}\omega_e q u_e^* u_m \mathbf{kx}$. (The zeroth order vanishes, as the bound states are orthogonal to the continuous spectrum.) We take the ground state u_0 as initial state u_m, and the final state u_e expanded as in (3.17). We thus find, by means of (3.19),

$$\int \rho_{mn}\,\mathrm{e}^{\mathrm{i}\mathbf{kx}}\,\mathrm{d}^3x \approx \frac{2\pi q\sqrt{2\omega_e}k\,\mathrm{e}^{\mathrm{i}\delta_1}}{k_e r_B L^{3/2}}\mathbf{k}_0\mathbf{k}_{e,0}\int_0^\infty u_0 R^*_{k_e,1} r^3\,\mathrm{d}r, \tag{3.24}$$

where $\mathbf{k}_{e,0}$ and $\mathbf{k}_0$ denote the electronic and tachyonic unit wave vectors. The integration is again standard, via (3.21) with $n = 1$. (On the right-hand side of (3.21), this amounts to putting $n = 0$ and to add a factor of $2r_B(1 + k_e^2 r_B^2)^{-1}$.) The longitudinal section (3.4) is assembled by substituting (3.24) together with (3.18) and (3.21) into the transition rate (3.3),

$$\mathrm{d}\sigma^{\mathrm{L}} \sim \frac{2^8\pi\alpha_q c^3 m m_t^2 r_B^4}{\hbar^3 k\left(1 + k_e^2 r_B^2\right)^5} b(k_e r_B)(\mathbf{k}_0\mathbf{k}_{e,0})^2\,\mathrm{d}\Omega. \tag{3.25}$$

The factor $b(x)$ is defined in (3.22), and we may rewrite this as, cf (3.23),

$$\mathrm{d}\sigma^{\mathrm{L}} \sim \frac{2^7\pi\alpha_q r_B^2 m_t^2 c^4 E_0^4 b(k_e r_B)}{(\hbar\omega)^5\sqrt{(\hbar\omega)^2 + (m_t c^2)^2}}(\mathbf{k}_0\mathbf{k}_{e,0})^2\,\mathrm{d}\Omega. \tag{3.26}$$

In the limit $k_e r_B \gg 1$, we recover the Born approximation (3.13), cf after (3.23). The angular parametrization in the differential sections (3.23) and (3.26) can be chosen as in (3.8) and (3.13). Thus the total cross sections $\sigma^{\mathrm{T,L}}$ are obtained by replacing $(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{e,0})^2\,\mathrm{d}\Omega$ in (3.23) and $(\mathbf{k}_0\mathbf{k}_{e,0})^2\,\mathrm{d}\Omega$ in (3.26) by a factor $4\pi/3$. At the ionization threshold, in the limit $\hbar\omega \to E_0$,

$$\sigma^{\mathrm{T}} \sim \frac{2^9\pi^2}{3\,\mathrm{e}^4}\frac{\alpha_q r_B^2 E_0}{\sqrt{E_0^2 + (m_t c^2)^2}}, \qquad \sigma^{\mathrm{L}} \sim \frac{2^9\pi^2}{3\,\mathrm{e}^4}\frac{\alpha_q r_B^2 (m_t c^2)^2}{E_0\sqrt{E_0^2 + (m_t c^2)^2}}, \tag{3.27}$$

where $\mathrm{e} \approx 2.718$ and $E_0/(m_t c^2) \approx Z^2/158$, cf after (2.31) and (3.8). This limit is studied in greater detail for Rydberg states in section 4. The electromagnetic counterpart to the transversal section is obtained by replacing $\alpha_q \to \alpha_e$ and putting the tachyon mass to zero, like in Born approximation.

Finally, the tachyonic recombination cross sections are found by balancing emission and absorption rates, $\sigma^{\mathrm{T}}_{\mathrm{rec}} = 2(k/k_e)^2\sigma^{\mathrm{T}}$ and $\sigma^{\mathrm{L}}_{\mathrm{rec}} = (k/k_e)^2\sigma^{\mathrm{L}}$, reflecting the symmetry of the Einstein coefficients in (2.23)–(2.26). The factor of 2 is the weight of two transversal degrees, and this symmetry extends as it stands to all s-states. More explicitly,

$$\sigma^{\mathrm{T}}_{\mathrm{rec}} = \frac{\hbar^2\omega^2 + m_t^2 c^4}{mc^2 E_e}\sigma^{\mathrm{T}}, \qquad \sigma^{\mathrm{L}}_{\mathrm{rec}} = \frac{\hbar^2\omega^2 + m_t^2 c^4}{2mc^2 E_e}\sigma^{\mathrm{L}}, \tag{3.28}$$

where E_e and $\hbar\omega = \left(1 + k_e^2 r_B^2\right)E_0$ are the energies of the incident electron and the emitted tachyon. These relations apply to both limits, Born and dipole, of course. The recombination cross sections refer to electron capture in the empty K-shell, irrespective of the spin. The ionization cross sections $\sigma^{\mathrm{T,L}}$ refer to a single electron in the K-shell. At the ionization threshold, we find the ratio $\sigma^{\mathrm{L}}/\sigma^{\mathrm{T}} \sim m_t^2 c^4 \big/ E_0^2 \approx 2.5 \times 10^4/Z^4$, as well as

$$\frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{ph}}} \sim \frac{q^2}{e^2}\frac{m_t c^2}{E_0\sqrt{1 + (E_0/m_t c^2)^2}}, \qquad \frac{\sigma^{\mathrm{L}}_{\mathrm{rec}}}{\sigma^{\mathrm{ph}}_{\mathrm{rec}}} \sim \frac{1}{2}\left(\frac{m_t^2 c^4}{E_0^2} + 1\right)\frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{ph}}}. \tag{3.29}$$

In the case of low charge numbers, $(Z^2/158)^2 \ll 1$, cf after (3.27), we estimate $\sigma^{\mathrm{L}}/\sigma^{\mathrm{ph}} \approx 2.2 \times 10^{-9}/Z^2$ and $\sigma^{\mathrm{L}}_{\mathrm{rec}}\big/\sigma^{\mathrm{ph}}_{\mathrm{rec}} \approx 2.76 \times 10^{-5}/Z^6$. These ratios are not very promising yet, but can be greatly improved by replacing the ground state by a highly excited s-state, as is done in the next section.

4. Tachyonic ionization of Rydberg states

We derive the tachyonic ionization cross sections of hydrogenic Rydberg states with zero angular momentum, s-states,

$$u_n = (2mc^2/\hbar)^{-1/2} R_{n0}(r),$$
$$R_{n0}(r) = \frac{1}{\sqrt{\pi}(r_{\rm B}n)^{3/2}}\, {\rm e}^{-r/(r_{\rm B}n)} \frac{1}{n} L_{n-1}^{(1)}\left(\frac{2r}{r_{\rm B}n}\right) \tag{4.1}$$

with n in the range 10^2–10^3 and beyond [19–23]. The $L_n^{(\alpha)}(x)$ are Laguerre polynomials, $L_0^{(\alpha)}(x) = 1$ and $L_{n-1}^{(1)}(x) = n\,_1F_1(1-n; 2; x)$. The normalization is $4\pi \int_0^\infty R_{n0}^2(r) r^2\, {\rm d}r = 1$, which follows from the orthogonality relation,

$$\int_0^\infty L_n^{(\alpha)}(r) L_m^{(\alpha)}(r)\, {\rm e}^{-r} r^\alpha\, {\rm d}r = \delta_{mn}\Gamma(\alpha+n+1)/\Gamma(n+1), \tag{4.2}$$

and the recursion relation $L_n^{(\alpha)} = L_n^{(\alpha+1)} - L_{n-1}^{(\alpha+1)}$. Hence, $\int_0^\infty L_{n-1}^{(1)2}(r)\, {\rm e}^{-r} r^2\, {\rm d}r = 2n^2$ and $\int_0^\infty L_{n-1}^{(1)2}(r)\, {\rm e}^{-r} r^3\, {\rm d}r = 6n^3$, so that the radial expectation value reads [19]

$$\langle r\rangle = 4\pi \int_0^\infty R_{n0}^2(r) r^3\, {\rm d}r = \tfrac{3}{2} r_{\rm B} n^2, \tag{4.3}$$

where $r_{\rm B}n^2$ is the semiclassical orbital radius.

The Born approximation of the ground state cross sections can easily be generalized to excited s-states; we only indicate the alterations in the formulae of section 3 to that effect. We take u_n in (4.1) as the initial state, which replaces u_m or u_0 in section 3. The integral $\int u_n\, {\rm e}^{{\rm i}(\mathbf{k}-\mathbf{k}_{\rm e})\mathbf{x}}\, {\rm d}^3x$, cf (3.7), is calculated via the Fourier transform

$$F_{\rm R}(k) := \int R_{n0}\, {\rm e}^{-{\rm i}\mathbf{kx}}\, {\rm d}^3x = \frac{2{\rm i}(-)^{n+1}\sqrt{\pi r_{\rm B}n}}{k(1+(kr_{\rm B}n)^2)}(y^n - y^{-n}), \qquad y := \frac{1-{\rm i}kr_{\rm B}n}{1+{\rm i}kr_{\rm B}n}, \tag{4.4}$$

where we used,

$$\int_0^\infty L_{n-1}^{(1)}\left(\frac{2r}{r_{\rm B}n}\right) {\rm e}^{-(1/(r_{\rm B}n)+{\rm i}k)r} r\, {\rm d}r = \frac{(-)^{n+1} r_{\rm B}^2 n^3}{1+(kr_{\rm B}n)^2} y^n. \tag{4.5}$$

We substitute $\mathbf{k} \to \mathbf{k} - \mathbf{k}_{\rm e}$ into (4.4), and expand in $k/k_{\rm e} \ll 1$, so that $|\mathbf{k}-\mathbf{k}_{\rm e}| \approx k_{\rm e}(1-(k/k_{\rm e})\cos\theta)$, as well as in $1/(k_{\rm e}r_{\rm B}n)$, arriving at a slightly changed equation (3.7). On the left-hand side of (3.7), u_0 is replaced by u_n, and on the right-hand side we rescale the Bohr radius, $r_{\rm B} \to r_{\rm B}n$, and add a factor of n. We thus have to multiply the transversal cross section (3.8) by a factor of n^2 and to replace the Bohr radius by $r_{\rm B}n$, which results in an overall rescaling by n^{-3}.

We turn to the longitudinal cross section. The generalization of the Fourier transform (3.11) is $(2mc^2/\hbar)^{-1/2} F_{\rm R}(k)$, cf (4.4). The matrix element (3.12) is calculated in Fourier space, with u_n in (4.1), via $\int u_1^* u_2\, {\rm e}^{{\rm i}\mathbf{kx}}\, {\rm d}^3x = (2\pi)^{-3} \int \hat{u}_1^*(\mathbf{q})\hat{u}_2(\mathbf{q}-\mathbf{k})\, {\rm d}^3q$; the convention for Fourier transforms is $\hat{u} = \int u {\rm e}^{-{\rm i}\mathbf{qx}}\, {\rm d}^3x$. We find, cf (3.12),

$$\int u_n {\rm e}^{{\rm i}\mathbf{kx}}\left({\rm e}^{-{\rm i}\mathbf{k}_{\rm e}\mathbf{x}} + \psi_{k_{\rm e}}^*\right) {\rm d}^3x = \int \frac{F_{\rm R}(|\mathbf{q}-\mathbf{k}|)}{(2mc^2/\hbar)^{1/2}} \left(\delta(\mathbf{q}-\mathbf{k}_{\rm e}) + \frac{\hat{\psi}_{k_{\rm e}}(\mathbf{q})}{(2\pi)^3}\right) {\rm d}^3q, \tag{4.6}$$

where $\hat{\psi}_{k_{\rm e}}$ is defined in (3.10). In the $F_{\rm R}\hat{\psi}$-term on the right-hand side, we approximate $F_{\rm R}$ by a δ-function,

$$\sqrt{\pi}(r_{\rm B}n)^{3/2} F_{\rm R}(|\mathbf{q}|) \to (2\pi)^3\delta(\mathbf{q}), \tag{4.7}$$

applicable for large $r_{\rm B}n$. (It is easy to see that (4.7) is a limit definition of the Dirac function, by means of the real space representation (4.4), the $1/(r_{\rm B}n)$-scaling of the Laguerre polynomial in (4.1), and the normalization $L_{n-1}^{(1)}(0) = n$.) The integrations in (4.6) are trivial, and we subsequently perform the approximations pointed out after (4.5), that is, expansions in $k/k_{\rm e} \ll 1$ and $1/(k_{\rm e} r_{\rm B} n) \ll 1$. We arrive at (3.12), with the right-hand side multiplied by n and $r_{\rm B}$ replaced by $r_{\rm B} n$, which amounts to an overall rescaling by $n^{-3/2}$. On the left-hand side of (3.12), the only change is $u_0 \to u_n$, cf (4.1). The longitudinal cross section (3.13) is thus recovered, rescaled by a factor of n^{-3}, like the transversal section discussed above.

As for the recombination cross sections, identities (3.28) apply to any principal quantum number as long as the electron gets caught in an s-state. Energy conservation implies $\hbar\omega = (1 + (k_{\rm e} r_{\rm B} n)^2) E_n$, where $E_n = \hbar^2 \big/ \big(2 m r_{\rm B}^2 n^2\big)$ is the ionization energy of the nth level, so that $E_n[\mathrm{Ry}] = Z^2/n^2$ with $2\,\mathrm{Ry} = 27.21$ eV, and ω is the excess energy carried by the emitted tachyon. Here, $r_{\rm B} = \hbar/(mc\alpha_Z)$ is the ground state Bohr radius with $\alpha_Z \approx Z/137$ as defined before (3.5), and $m_{\rm t} c^2 \approx 158$ Ry, cf (3.27). In Born approximation, $k_{\rm e} r_{\rm B} n \gg 1$, we may neglect the ionization energy and approximate $\hbar\omega \approx E_{\rm e}$ in (3.28). Moreover, the n^{-3}-scaling of the cross sections also applies to small n, and capture in s-states is always dominant [22, 23]. The total recombination cross sections, obtained by summation over all s-states, are thus given by (3.28), with (3.14) substituted and multiplied by $\zeta(3) \approx 1.202$.

The dipole approximation is harder to deal with, but it gives access to the ionization threshold, which is the most likely place in momentum space to find superluminal quanta, cf the discussion after (4.31). We again take the calculation of the ground state cross sections as the starting point, and proceed by pointing out the modifications required by excited s-states. The technical changes needed for large quantum numbers get quite extensive, due to the high-order polynomials occurring in the matrix elements; asymptotic approximations to these polynomials are derived in the appendix.

We start with the transversal matrix element, $\int \boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \mathbf{j}_{mn}\, \mathrm{d}^3x$, cf (3.20). In the integral on the right-hand side of (3.20), we have to replace u_0 by $-r_{\rm B} u_n'$, cf (3.16), where we use $\nabla R_{n0} = R_{n0}' \mathbf{r}_0$ when replacing the initial state u_m by u_n in (4.1). In the longitudinal element (3.24), we just have to replace u_0 by u_n to obtain the dipole approximation of $\int \rho_{mn}\, \mathrm{e}^{\mathrm{i}\mathbf{kx}}\, \mathrm{d}^3x$. These matrix elements, (3.20) and (3.24) (with the indicated changes), can be reduced to integrals [24]

$$\int_0^\infty r^3\, \mathrm{d}r\, \mathrm{e}^{-(1/(r_{\rm B} n) - \mathrm{i}k_{\rm e})r}\, {}_1F_1(m; 4; 2r/(r_{\rm B} n))\, {}_1F_1(2 - \mathrm{i}\rho; 4; -2\mathrm{i}k_{\rm e} r)$$
$$= \frac{6(r_{\rm B} n)^4 (-)^m}{(1 + (k_{\rm e} r_{\rm B} n)^2)^2}\, y^{m - \mathrm{i}\rho}\, {}_2F_1(m, 2 - \mathrm{i}\rho; 4; 1 - y^2), \tag{4.8}$$

with (negative) integer m. Here, we have introduced the shortcuts,

$$y = \frac{1 - \mathrm{i}k_{\rm e} r_{\rm B} n}{1 + \mathrm{i}k_{\rm e} r_{\rm B} n}, \qquad 1 - y^2 = \frac{4\mathrm{i}k_{\rm e} r_{\rm B} n}{(1 + \mathrm{i}k_{\rm e} r_{\rm B} n)^2}, \qquad \rho = \frac{1}{k_{\rm e} r_{\rm B}}. \tag{4.9}$$

These integrals are real; to see this, we put $z = 1 - y^2$ and note the symmetries $z^* = z/(z-1)$ and $y^* = y^{-1}$, as well as

$${}_2F_1(m, \beta; 4; z) = (1 - z)^{-m}\, {}_2F_1(m, 4 - \beta; 4; z/(z - 1)). \tag{4.10}$$

Moreover, $y^{m - \mathrm{i}\sigma} = \mathrm{e}^{-2(\sigma + \mathrm{i}m)\arctan(n/\rho)}$, with real σ and principal values like in (3.22); the expansions of $\arctan x$ read $x - x^3/3 + \cdots$ and $\pi/2 - 1/x + \cdots$. For the integrals (4.8) to be applicable, we first have to express the Laguerre polynomials occurring in the matrix elements (4.17) and (4.19) in terms of $L_k^{(3)}$, and then use

$$L_k^{(3)}(x) = \tfrac{1}{6}(k + 3)(k + 2)(k + 1)\, {}_1F_1(-k; 4; x), \tag{4.11}$$

where $k \geqslant -3$. The polynomial $L_{n-1}^{(1)}$ and its derivative, $\mathrm{d}L_{n-1}^{(1)}/\mathrm{d}x = -L_{n-2}^{(2)}$, as well as $xL_{n-1}^{(1)}$, can be written as linear combinations of $L_k^{(3)}$, by means of two recursion relations. One was indicated after (4.2) and gives

$$\begin{aligned} &L_{n-1}^{(1)} = L_{n-1}^{(3)} - 2L_{n-2}^{(3)} + L_{n-3}^{(3)} = L_{n-1}^{(4)} - 3L_{n-2}^{(4)} + 3L_{n-3}^{(4)} - L_{n-4}^{(4)}, \\ &\mathrm{d}L_{n-1}^{(1)}\big/\mathrm{d}x = -\big(L_{n-2}^{(3)} - L_{n-3}^{(3)}\big). \end{aligned} \tag{4.12}$$

The second type of recursion needed is $xL_n^{(\alpha+1)} = (n+\alpha+1)L_n^{(\alpha)} - (n+1)L_{n+1}^{(\alpha)}$, which gives, in combination with (4.12),

$$xL_{n-1}^{(1)} = -n\big(L_n^{(3)} - 4L_{n-1}^{(3)} + 6L_{n-2}^{(3)} - 4L_{n-3}^{(3)} + L_{n-4}^{(3)}\big). \tag{4.13}$$

We use the shortcut,

$$F_m := {}_2F_1(m, 2-\mathrm{i}\rho; 4; 1-y^2), \qquad \rho = 1/(k_\mathrm{e} r_\mathrm{B}), \tag{4.14}$$

and the hypergeometric contiguous relations [24],

$$\begin{aligned} &(n-2)y^2F_{3-n} = -(n+2)F_{1-n} + [(n-\mathrm{i}\rho)y^2 + (n+\mathrm{i}\rho)]F_{2-n}, \\ &(n+3)F_{-n} = [(n+1-\mathrm{i}\rho)y^2 + (n+1+\mathrm{i}\rho)]F_{1-n} - (n-1)y^2F_{2-n}. \end{aligned} \tag{4.15}$$

By iterating the first identity (with n and $n-1$), we find

$$\begin{aligned} (n-2)(n-3)y^4F_{4-n} &= -(n+2)[(n-1-\mathrm{i}\rho)y^2 + (n-1+\mathrm{i}\rho)]F_{1-n} \\ &\quad + [a_0^* y^4 + (n^2 - n + 2 + 2\rho^2)y^2 + a_0]F_{2-n}, \end{aligned} \tag{4.16}$$

where $a_0 := (n+\mathrm{i}\rho)(n-1+\mathrm{i}\rho)$. The subsequent matrix elements are linear combinations of F_{1-n} and F_{2-n}, which are polynomials for negative integer index.

The transversal element (3.20) (with the modification explained before (4.8)) reads

$$\begin{aligned} R^\mathrm{T} &:= \int_0^\infty R'_{n,0} R^*_{k_\mathrm{e},1} r^2\,\mathrm{d}r = -\frac{2\sqrt{2k_\mathrm{e}\big(1+k_\mathrm{e}^2 r_\mathrm{B}^2\big)}}{3r_\mathrm{B}^3 n^{7/2}\sqrt{1-\mathrm{e}^{-2\pi\rho}}} \\ &\quad\times \int_0^\infty r^3\,\mathrm{d}r\,\mathrm{e}^{-(1/(r_\mathrm{B}n)-\mathrm{i}k_\mathrm{e})r}\big(L_{n-1}^{(3)} - L_{n-3}^{(3)}\big)\left(\frac{2r}{r_\mathrm{B}n}\right){}_1F_1(2-\mathrm{i}\rho; 4; -2\,\mathrm{i}k_\mathrm{e}r), \end{aligned} \tag{4.17}$$

where we made use of (4.12). The radial wavefunctions $R_{n,0}$ and $R_{k_\mathrm{e},1}$ are defined in (4.1) and (3.18), respectively. By virtue of (4.11) and (4.8)

$$\begin{aligned} &R^\mathrm{T} = A(2n(n+2)F_{1-n} + (n-1)pF_{2-n}), \\ &p := (-n+\mathrm{i}\rho)y^2 - (n+\mathrm{i}\rho) = 2n(\rho^2+n^2)(\rho+\mathrm{i}n)^{-2}, \\ &A := \frac{2r_\mathrm{B}\sqrt{2k_\mathrm{e}\big(1+k_\mathrm{e}^2r_\mathrm{B}^2\big)}}{3\sqrt{1-\mathrm{e}^{-2\pi\rho}}}\,\frac{(-)^n n^{3/2} y^{1-n-\mathrm{i}\rho}}{(1+(k_\mathrm{e}r_\mathrm{B}n)^2)^2}, \qquad y = \frac{\rho-\mathrm{i}n}{\rho+\mathrm{i}n}, \end{aligned} \tag{4.18}$$

where F_m is defined in (4.14), y in (4.9), and $\rho = 1/(k_\mathrm{e}r_\mathrm{B})$. After the integration (4.8), we used the first recursion relation in (4.15).

The longitudinal matrix element (3.24) is modified as explained before (4.8),

$$\begin{aligned} R^\mathrm{L} &:= \int_0^\infty R_{n,0} R^*_{k_\mathrm{e},1} r^3\,\mathrm{d}r = \frac{\sqrt{2k_\mathrm{e}\big(1+k_\mathrm{e}^2r_\mathrm{B}^2\big)}}{3r_\mathrm{B}n^{3/2}\sqrt{1-\mathrm{e}^{-2\pi\rho}}} \\ &\quad\times \int_0^\infty r^3\,\mathrm{d}r\,\mathrm{e}^{-(1/(r_\mathrm{B}n)-\mathrm{i}k_\mathrm{e})r}\,\frac{2r}{r_\mathrm{B}n}L_{n-1}^{(1)}\left(\frac{2r}{r_\mathrm{B}n}\right){}_1F_1(2-\mathrm{i}\rho; 4; -2\mathrm{i}k_\mathrm{e}r). \end{aligned} \tag{4.19}$$

Here, we substitute (4.13), and perform the integration by means of (4.11) and (4.8). We then employ the recursion relations (4.15) and (4.16) to arrive at

$$\begin{aligned} R^{\mathrm{L}} &= -\frac{r_{\mathrm{B}}^2 n^2 A}{2y}((n+2)q_1 F_{1-n} + (n-1)q_2 F_{2-n}), \\ q_1 &:= 2(2n - \mathrm{i}\rho)y^2 + 8ny + 2(2n + \mathrm{i}\rho) = 8n\rho^2(\rho + \mathrm{i}n)^{-2}, \\ q_2 &:= a_0^* y^4 + 4n(n - \mathrm{i}\rho)y^3 + 2(3n^2 + n + \rho^2)y^2 + 4n(n + \mathrm{i}\rho)y + a_0 \\ &= 8n\rho^2(\rho^2 + n^2)(\rho + \mathrm{i}n)^{-4}, \end{aligned} \tag{4.20}$$

where A, y and F_m are defined in (4.18) and (4.14), and a_0 in (4.16).

Approximations of the hypergeometric polynomials F_{1-n} and F_{2-n} are indispensable even for moderate n, as demonstrated by the exponential in (4.22). The hypergeometric F_m in (4.14) contracts to a confluent function in the limit $\rho \to \infty$ (with the other parameters fixed), according to the limit definition ${}_2F_1(a, b; c; z/b) \to {}_1F_1(a; c; z)$ for $|b| \to \infty$. Hence, $F_m \to {}_1F_1(m; 4; 4n)$, for fixed m and n, since $1 - y^2 \sim 4\mathrm{i}n/\rho$ if $n/\rho \ll 1$. We do not need uniformity here, as no further integrations are necessary to find the cross sections. The confluent hypergeometric function so obtained is a Laguerre polynomial, cf (4.11). Performing the limit $\rho \to \infty$ in the matrix elements (4.18) and (4.20), we find

$$\begin{aligned} R^{\mathrm{T}} &\sim 12A(n+1)^{-1}\big(L_{n-1}^{(3)}(4n) + L_{n-2}^{(3)}(4n)\big), \\ R^{\mathrm{L}} &\sim -2r_{\mathrm{B}}^2 n^2 R^{\mathrm{T}}, \qquad A \sim (2^{3/2}/3)r_{\mathrm{B}} k_{\mathrm{e}}^{1/2}(-)^n n^{3/2}\,\mathrm{e}^{-2n}. \end{aligned} \tag{4.21}$$

The exponential in A, cf (4.18), stems from the expansion of the arctan stated after (4.10). An Airy approximation of Laguerre polynomials was derived in [25], from which we infer,

$$L_{n-1}^{(3)}(4n) + L_{n-2}^{(3)}(4n) = \frac{2^{1/3}(-)^{n-1}\,\mathrm{e}^{2n}}{8 \times 3^{1/3}\Gamma(1/3)n^{2/3}}(1 + \mathrm{O}(n^{-2/3})). \tag{4.22}$$

Hence, in the limit $\rho \to \infty$ and for large n,

$$R^{\mathrm{T}} \sim -\frac{2^{5/6} r_{\mathrm{B}} k_{\mathrm{e}}^{1/2} n^{-1/6}}{3^{1/3}\Gamma(1/3)}, \tag{4.23}$$

and R^{L} follows from (4.21).

The opposite limit, $\rho \ll 1$, with n fixed, can be settled by the ascending series expansion of the hypergeometric functions in (4.18) and (4.20), since $1 - y^2 \sim -4\mathrm{i}\rho/n$. In leading order, it is sufficient to put $F_{1-n} \sim F_{2-n} \sim 1$ in the matrix elements, so that

$$R^{\mathrm{T}} \sim -4\pi^{-1/2} r_{\mathrm{B}} k_{\mathrm{e}}^{1/2} \rho^{5/2} n^{-3/2}, \qquad R^{\mathrm{L}} \sim 8\pi^{-1/2} r_{\mathrm{B}}^3 k_{\mathrm{e}}^{1/2} \rho^{9/2} n^{-3/2}. \tag{4.24}$$

In the approximations (4.21) and (4.24), there is no expansion in the quantum number n involved. In (4.24), the asymptotics is based on $\rho \ll 1$, so that higher orders can be obtained from the hypergeometric series. In (4.21), the limit $\rho \to \infty$ is actually carried out.

The limit $\rho/n \ll 1$ with $\rho \gg 1$ and the limit $n/\rho \ll 1$ with $n \gg 1$ amount to asymptotic expansions of the hypergeometric polynomials in (4.18) and (4.20) with two large parameters. The steepest descent approximation of $R^{\mathrm{T,L}}$ for large but finite ρ and n is derived in the appendix, cf (A.27) and (A.28). The squared matrix elements read

$$|R^{\mathrm{T}}|^2 = \frac{2^{5/3} r_{\mathrm{B}}^2 k_{\mathrm{e}} \alpha_\infty^{8/3} X}{3^{2/3}\Gamma^2(1/3) n^{1/3}\big(1 + \alpha_\infty^2\big)^{4/3}}, \qquad |R^{\mathrm{L}}|^2 = \frac{2^{11/3} r_{\mathrm{B}}^6 k_{\mathrm{e}} n^{11/3} \alpha_\infty^{20/3} X}{3^{2/3}\Gamma^2(1/3)\big(1 + \alpha_\infty^2\big)^{10/3}}, \tag{4.25}$$

where the ratio $\alpha_\infty := \rho/n$ is kept fixed, and

$$X := 1 + \frac{2^{5/6}\Gamma(1/3)\sqrt{\alpha_\infty + \sqrt{\alpha_\infty^2 + 4}}}{3^{1/3}\Gamma(2/3)(\alpha_\infty n)^{1/6}\left(1 + \alpha_\infty^2\right)^{1/6}} + \frac{2^{2/3}\Gamma^2(1/3)\sqrt{\alpha_\infty^2 + 4}}{3^{2/3}\Gamma^2(2/3)(\alpha_\infty n)^{1/3}\left(1 + \alpha_\infty^2\right)^{1/3}} + \mathrm{O}(n^{-1/2}, \rho^{-1/2}). \quad (4.26)$$

In the limit $\alpha_\infty \to \infty$, we recover $R^{\mathrm{T,L}}$ in (4.23) and (4.21), including the next two orders. (The asymptotics in the appendix is not really designed for this, nevertheless this limit can be carried out in the matrix elements.) Expansion (4.26) is also valid for small α_∞, provided $\rho = \alpha_\infty n$ stays large.

We turn to the cross sections. The squared current matrix elements in (3.2) are compiled from (3.20), and the elements (3.3) of the charge density are obtained from (3.24), both modified as explained before (4.8),

$$\left|\int \varepsilon_{\mathbf{k},\lambda}\mathbf{j}_{mn}\, \mathrm{e}^{\mathrm{i}\mathbf{kx}}\, \mathrm{d}^3x\right|^2 \sim \frac{16\pi^3\alpha_{\mathrm{q}}\hbar^3 c}{L^3 m^2 r_{\mathrm{B}}^2 k_{\mathrm{e}}^2}|R^{\mathrm{T}}|^2(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0})^2,$$
$$\left|\int \rho_{mn}\, \mathrm{e}^{\mathrm{i}\mathbf{kx}}\, \mathrm{d}^3x\right|^2 \sim \frac{16\pi^3\alpha_{\mathrm{q}}\hbar c k^2}{L^3 r_{\mathrm{B}}^2 k_{\mathrm{e}}^2}|R^{\mathrm{L}}|^2(\mathbf{k}_0\mathbf{k}_{\mathrm{e},0})^2. \quad (4.27)$$

The matrix elements $R^{\mathrm{T,L}}$ are stated in (4.18) and (4.20), in terms of hypergeometric polynomials, and α_{q} is the tachyonic fine structure constant, cf (3.8). We thus arrive at, cf (3.2)–(3.4),

$$\mathrm{d}\sigma_\lambda^{\mathrm{T}} \sim \frac{2\pi\alpha_{\mathrm{q}}\hbar}{mc r_{\mathrm{B}}^2 k_{\mathrm{e}} k}|R^{\mathrm{T}}|^2(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0})^2\, \mathrm{d}\Omega,$$
$$\mathrm{d}\sigma^{\mathrm{L}} \sim \frac{2\pi\alpha_{\mathrm{q}} c^3 m_{\mathrm{t}}^2 m}{\hbar^3 r_{\mathrm{B}}^2 k_{\mathrm{e}} k}|R^{\mathrm{L}}|^2(\mathbf{k}_0\mathbf{k}_{\mathrm{e},0})^2\, \mathrm{d}\Omega. \quad (4.28)$$

In the following, we study limit cases, based on the approximations (4.21)–(4.26) of $R^{\mathrm{T,L}}$.

First, as a consistency check, we recover from (4.18)–(4.24) the cross sections previously derived. We invoke energy conservation at the nth level, cf after (4.7), as well as the tachyonic dispersion relation stated before (3.23). The limit $\rho \to \infty$ with $n = 1$ in (4.21) can be traced back to the ground state ionization cross sections (3.22) and (3.25), if we replace there $b(k_{\mathrm{e}} r_{\mathrm{B}})$ by its limit value e^{-4}. The Born approximation, the limit $\rho \to 0$ with arbitrary n, is recovered from (4.24). (The transversal section was derived in (3.8) for ground state ionization, and its n^3-scaling for general s-states is explained after (4.5). The n^3-scaling of the longitudinal section (3.13) was derived after (4.7).) Finally, the cross sections (4.28) for $n = 1$ and arbitrary ρ can directly be obtained from $R^{\mathrm{T,L}}$ in (4.18) and (4.20), since $F_0 = 1$ and F_1 drops out. In this way, the ground state ionization (3.22) and (3.25) is recovered for finite ρ.

The matrix elements $R^{\mathrm{T,L}}$ in (4.18) and (4.20) are polynomials and can be used in the cross sections (4.28) without approximations, as long as the quantum number n stays small. Rydberg states do not qualify in this respect, cf the exponential in (4.22), which takes us to the most interesting limit, $\rho \to \infty$ and $n \gg 1$, cf (4.23), accounting for highly excited states just ionized so that the ejected electron is free, but barely so, with zero momentum. In this limit, at the ionization threshold, the cross sections (4.28) simplify to

$$\mathrm{d}\sigma_\lambda^{\mathrm{T}} \sim \frac{\kappa_0\alpha_{\mathrm{q}}\hbar^2(\varepsilon_{\mathbf{k},\lambda}\mathbf{k}_{\mathrm{e},0})^2\, \mathrm{d}\Omega}{m\sqrt{(\hbar\omega)^2 + (m_{\mathrm{t}}c^2)^2}}, \qquad \kappa_0 := \frac{2^{2/3}4\pi}{3^{2/3}\Gamma^2(1/3)n^{1/3}}, \quad (4.29)$$

$$\mathrm{d}\sigma^{\mathrm{L}} \sim \frac{m_{\mathrm{t}}^2 c^4}{E_n^2} \frac{\kappa_0 \alpha_{\mathrm{q}} \hbar^2 (\mathbf{k}_0 \mathbf{k}_{\mathrm{e},0})^2 \, \mathrm{d}\Omega}{m\sqrt{(\hbar\omega)^2 + (m_{\mathrm{t}} c^2)^2}}, \qquad \frac{1}{E_n^2} = 4(r_{\mathrm{B}} n)^4 \frac{m^2}{\hbar^4}. \tag{4.30}$$

This is the leading order, the two next-to-leading orders are obtained by adding the factor X in (4.26) with $\alpha_\infty = \infty$. This limit smoothly extends to finite electron energy, $\rho \gg 1$, by a rescaling of the constant κ_0 in the cross sections

$$\kappa^{\mathrm{T}} := \frac{\kappa_0 \alpha_\infty^{8/3} X}{\left(1 + \alpha_\infty^2\right)^{4/3}}, \qquad \kappa^{\mathrm{L}} := \frac{\alpha_\infty^4}{\left(1 + \alpha_\infty^2\right)^2} \kappa^{\mathrm{T}}, \tag{4.31}$$

which means to replace κ_0 in (4.29) by κ^{T} and in (4.30) by κ^{L}. The factors $\kappa^{\mathrm{T,L}}$ can be read off from the squared $R^{\mathrm{T,L}}$ in (4.25), where $\alpha_\infty = \rho/n$ and X stands for expansion (4.26). The applicability of cross sections (4.29)–(4.31) hinges upon $\rho = 1/(k_{\mathrm{e}} r_{\mathrm{B}}) \gg 1$, which is tantamount to $E_{\mathrm{e}} \ll E_{n=1}$, where $E_{\mathrm{e}} = \hbar^2 k_{\mathrm{e}}^2 \big/ (2m)$ is the energy of the ejected electron and $E_{n=1}$ is the ground state ionization energy, cf after (4.7).

Cross sections (4.29) and (4.30) account for ionization in the limit $\rho \to \infty, n \gg 1$. This limit, the ionization threshold, is of crucial importance for the detection of longitudinal radiation, as the longitudinal section overpowers the transversal one due to the n-scaled Bohr radius, cf (4.30). We show that the resulting n^4-factor in the longitudinal section has a marked impact on the ionization at Rydberg levels of order $n \approx 10^3$, as it counteracts the very small ratio of tachyonic and electric fine structure constants in the cross section ratios. The total cross sections $\sigma^{\mathrm{T,L}}$ are obtained by replacing $(\varepsilon_{\mathbf{k},\lambda} \mathbf{k}_{\mathrm{e},0})^2 \, \mathrm{d}\Omega$ and $(\mathbf{k}_0 \mathbf{k}_{\mathrm{e},0})^2 \, \mathrm{d}\Omega$ in (4.29) and (4.30) by a factor of $4\pi/3$. The total photonic cross section σ^{ph} is recovered from the transversal section in the limit of zero tachyon mass, by identifying tachyonic with electric charge as was done in (3.15). We note the ratio of the tachyonic and electric fine structure constants, $q^2/e^2 \approx 1.4 \times 10^{-11}$, as well as the tachyon–electron mass ratio, $m_{\mathrm{t}}/m \approx 1/238$, cf after (2.31) and (3.8). The cross section ratios, based on (4.29) and (4.30), read accordingly,

$$\frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{T}}} \sim \frac{m_{\mathrm{t}}^2 c^4}{E_n^2} \approx 2.5 \times 10^4 \frac{n^4}{Z^4}, \qquad \frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{ph}}} \sim \frac{q^2}{e^2} \frac{m_{\mathrm{t}} c^2}{E_n} \approx 2.2 \times 10^{-9} \frac{n^2}{Z^2}. \tag{4.32}$$

These ratios only apply at the ionization threshold, $k_{\mathrm{e}} = 0$, but can easily be extended to finite electronic momentum, $k_{\mathrm{e}} \ll 1/r_{\mathrm{B}}$, by virtue of, cf (4.31),

$$\frac{\kappa^{\mathrm{L}}}{\kappa^{\mathrm{T}}} = \frac{1}{(1 + (k_{\mathrm{e}} r_{\mathrm{B}} n)^2)^2} = \frac{E_n^2}{\hbar^2 \omega^2}, \tag{4.33}$$

where E_n is the ionization energy at the nth level, cf after (4.7). Subject to this relation, the ratios (4.32) extend to finite k_{e} as

$$\frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{T}}} \sim \frac{m_{\mathrm{t}}^2 c^4}{\hbar^2 \omega^2}, \qquad \frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{ph}}} \sim \frac{q^2}{e^2} \frac{m_{\mathrm{t}} c^2}{\hbar\omega}, \tag{4.34}$$

where the tachyonic energy $\omega(k_{\mathrm{e}}, n)$ is determined by the second equality in (4.33).

The velocity of the ionizing tachyon, inferred from

$$E_n + E_{\mathrm{e}} = \frac{m_{\mathrm{t}} c^2}{\sqrt{\upsilon^2/c^2 - 1}}, \tag{4.35}$$

is another reason why cross sections (4.29) and (4.30) are of special interest. The energetically minimal ionization of Rydberg states at the threshold $E_{\mathrm{e}} = 0$ (in practice, for $E_{\mathrm{e}} \ll E_n$ or $k_{\mathrm{e}} \ll 1/(r_{\mathrm{B}} n)$, cf after (4.31)) is effected by very speedy tachyons,

$$\frac{\upsilon}{c} \sim \frac{m_{\mathrm{t}} c^2}{E_n} \approx 158 \frac{n^2}{Z^2}. \tag{4.36}$$

This, together with the cross section ratio $\sigma^{\mathrm{L}}/\sigma^{\mathrm{ph}}$ in (4.32), makes Rydberg states a serious contender for the detection of the longitudinal radiation. Moreover, as pointed out after (3.13), the angular extrema of the transversal and longitudinal differential sections can be used to discriminate photons and longitudinal tachyons. The angular parametrization of the ground state sections, cf after (3.8) and (3.13), also applies to cross sections (4.28)–(4.30).

Another striking result relates to recombination at the ionization threshold. The identities (3.28) connecting ionization and recombination cross sections remain valid for angular independent excited states. On the basis of ionization ratios (4.32),

$$\frac{\sigma^{\mathrm{L}}_{\mathrm{rec}}}{\sigma^{\mathrm{ph}}_{\mathrm{rec}}} \sim \frac{m_{\mathrm{t}}^2 c^4}{2E_n^2}\frac{\sigma^{\mathrm{L}}}{\sigma^{\mathrm{ph}}} \approx 2.76 \times 10^{-5}\frac{n^6}{Z^6}, \tag{4.37}$$

so that already at $n \sim 10$ (and low Z) recombination with thermal electrons is more likely to be accompanied by the emission of longitudinal quanta than photons. The reverse process, ionization, can be used to detect this radiation directly, though in this case quantum numbers of order $n \sim 10^4$ are needed to achieve a moderate $\sigma^{\mathrm{L}}/\sigma^{\mathrm{ph}}$ ratio.

5. Conclusion

We have given a quantitative discussion of radiative transitions effected by superluminal quanta. Induced and spontaneous transitions between hydrogenic bound states, as well as tachyonic ionization and recombination cross sections, were investigated. The subluminal particles were treated as scalar and mostly non-relativistic, but the tachyonic 4-potential can also be coupled to Dirac spinors by minimal substitution, similarly as done in section 2 for the Klein–Gordon field [18]. As for relativistic ejection energies, the longitudinal relativistic cross section needs to be calculated from scratch, but the transversal section is obtained by a simple rescaling of the relativistic photonic cross section. In fact, the only change necessary for tachyonic γ-rays [26] or even high-energy x-rays is a rescaling with the ratio $\alpha_{\mathrm{q}}/\alpha_{\mathrm{e}}$ of the tachyonic and electric fine structure constants, since the tachyon mass drops out in the dispersion relation at γ-ray energies. We have already seen this rescaling in section 3, though we were mainly interested in the effect of the tachyon mass in the low-energy, soft x-ray regime, where the tachyon mass dominates the shape of the transversal section. The longitudinal cross section, however, depends in any regime on the tachyon mass and vanishes in the zero-mass limit [27].

The angular dependence of cross sections is perhaps the most practical means to disentangle transversal and longitudinal radiation. This has been demonstrated here with ionization cross sections, where the transversal angular maximum corresponds to the longitudinal minimum. One may expect that the differential cross sections of tachyonic Compton scattering can also be used to that effect; there should be a transversal and longitudinal tachyonic counterpart to the Klein–Nishina formula, pertinent to the acceleration of the electron by the incoming tachyonic wave field, triggering electromagnetic radiation. The tachyonic Thomson cross section, the non-relativistic classical limit, was already derived in [14], but a quantum mechanical version is still lacking, especially if the incident tachyonic x-rays have energies close to the tachyon mass. Another interesting cross section to be scrutinized is the conversion of tachyonic γ-rays into electron–positron pairs. Pair production by tachyons has not been studied in any limit and context as yet, e.g. in a Coulomb potential or strong magnetic field. I surmise that the cross section for the conversion of transversal tachyonic γ-rays is just the Bethe–Heitler cross section rescaled with the ratio $\alpha_{\mathrm{q}}/\alpha_{\mathrm{e}}$, as above.

Other mechanisms for the detection of longitudinal radiation modes have been suggested with respect to a finite photon mass (i.e. a positive mass-squared), such as a capacitor in a perfectly conducting shell, impenetrable for transversal waves [9]. This line of reasoning, focused on macroscopic current distributions, is unlikely to be applicable to tachyons. At least, there is no obvious tachyonic counterpart to a perfectly conducting shell or the skin depth of a conductor, or even to a macroscopic charge density, due to averaging effects caused by periodic sign changes of the tachyon potential [10].

The decisive advantage of Rydberg states is the n-scaling of the longitudinal cross section, which counteracts the very small ratio of electric and tachyonic fine structure constants. In recombination, at the ionization threshold, that is, for the capture of very low energy thermal electrons, the tachyonic emission is overwhelmingly longitudinal, and it already starts to outpace electromagnetic emission at $n \sim 10$, provided the charge number is kept low. A more direct detection mechanism is longitudinal ionization, where the cross section starts to compete with the electromagnetic counterpart at about $n \sim 10^4$, cf (4.32). Up to now, there is interstellar and increasingly laboratory evidence for Rydberg states of one order less [21]. The angular variation of the ionization cross sections is the same for all s-states, irrespectively of the quantum number. It is, however, strongly affected by the polarization of the ionizing radiation and can be used to distinguish longitudinal tachyons from photons.

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Appendix: Asymptotics of high-order hypergeometric polynomials

The matrix elements (4.18) and (4.20) are composed of hypergeometric polynomials depending on two large parameters. We derive asymptotic approximations in steepest descent, based on the integral representation [28, 29],

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)(1 - \mathrm{e}^{-2\pi \mathrm{i} a})^{-1}}{\Gamma(a)\Gamma(c - a)} \oint t^{a-1}(1 - t)^{c-a-1}(1 - tz)^{-b}\, \mathrm{d}t, \tag{A.1} $$

where c is integer, and the integration path is taken as an anticlockwise loop encircling the points $t = 0$ and $t = 1$ and excluding $t = 1/z$. (This can be checked by expanding in z and invoking a standard integral representation of the B-function.) We study a special case where the parameters a and b get simultaneously large, that is, linear combinations of the hypergeometric polynomials, cf (4.14),

$$ F_m := {}_2F_1(m, 2 - \mathrm{i}\rho; 4; z), \qquad z = \frac{4\mathrm{i}n\rho}{(\rho + \mathrm{i}n)^2}, \tag{A.2} $$

where $\rho \gg 1$ (real) and $m = k - n$, with $k = 1, 2$ (or at least $k \ll n$) and integer $n \gg 1$. The goal is to find a workable approximation of these high-order polynomials, for large n and ρ, keeping the ratio $\alpha_\infty = \rho/n$ fixed.

To this end, we reparametrize (A.2), defining α by $2 - \mathrm{i}\rho = \mathrm{i}\alpha(k - n)$, so that $\alpha \to \alpha_\infty$ for $n, \rho \to \infty$. The argument z in the hypergeometric function can be expanded in the small parameter ε defined by

$$ \varepsilon = \sqrt{\frac{\alpha k + 2\mathrm{i}}{n\alpha(1 + \alpha^2)}}, \qquad \alpha = \frac{\rho + 2\mathrm{i}}{n - k}, \tag{A.3} $$

where $\mathrm{Re}\,\varepsilon > 0$,

$$z = \frac{-4\mathrm{i}\alpha}{(1-\mathrm{i}\alpha)^2}[1-\varepsilon^2(1+\mathrm{i}\alpha)^2(1-\varepsilon^2\mathrm{i}\alpha(2+\mathrm{i}\alpha)+\cdots)]. \tag{A.4}$$

We always assume $n \gg 1$ and $\rho \gg 1$, but we will not make any assumption about the size of the ratio $\alpha_\infty = \rho/n$. If α_∞ is moderate or large, the actual expansion in (A.4) (and in the subsequent asymptotics) is in $\alpha_\infty\varepsilon \sim n^{-1/2}$; if $\alpha_\infty \ll 1$, the expansion parameter is $\varepsilon \sim \rho^{-1/2}$. By definition, ε is small if ρ and n are both large, irrespectively of their ratio α_∞ (which is kept fixed), and $\alpha_\infty\varepsilon$ is small even if α_∞ is large.

In (A.1), we identify $a = 2-\mathrm{i}\rho$ and $b = k-n$,

$$F_{k-n} = \frac{3\mathrm{e}^{\pi\rho}}{\pi\rho(1+\rho^2)}\oint t^{1-\mathrm{i}\rho}(1-t)^{1+\mathrm{i}\rho}(1-tz)^{n-k}\,\mathrm{d}t, \tag{A.5}$$

and split the integrand into a fast and a slowly varying factor, $\mathrm{e}^{n\log f(t)}g(t)$, where

$$f(t) = t^{-\mathrm{i}\alpha}(1-t)^{\mathrm{i}\alpha}(1-tz), \qquad g(t) = t^{-1+\mathrm{i}\alpha k}(1-t)^{3-\mathrm{i}\alpha k}(1-tz)^{-k}. \tag{A.6}$$

The saddle point is found by solving $f'(t_0) = 0$ or $1-t_0z = \mathrm{i}z\alpha^{-1}t_0(1-t_0)$,

$$t_0 = \frac{1}{2}(1-\mathrm{i}\alpha)\left[1-\mathrm{i}(1+\mathrm{i}\alpha)\varepsilon\left(1+\frac{\varepsilon^2}{2}\right)+\mathrm{O}(\varepsilon^4)\right], \tag{A.7}$$

with $\mathrm{Re}\,\varepsilon > 0$. (There is a second solution obtained by changing the sign of ε, but only t_0 is a saddle point.) We expand,

$$\log f(t) = \log f_0 + \varphi(t-t_0)+\cdots, \qquad \varphi(t) := \tfrac{1}{2}t^2 f_0''/f_0 + \tfrac{1}{6}t^3 f_0'''/f_0, \tag{A.8}$$

where the subscript zeros indicate the argument taken at t_0,

$$\begin{aligned}
\log f_0 &= (1+\mathrm{i}\alpha)\log\frac{1+\mathrm{i}\alpha}{1-\mathrm{i}\alpha} - 2\mathrm{i}\alpha(1+\mathrm{i}\alpha)\varepsilon^2 + \mathrm{O}(\varepsilon^4),\\
\frac{f_0''}{f_0} &= \frac{\mathrm{i}\alpha(1-\mathrm{i}\alpha-2t_0)}{t_0^2(1-t_0)^2} = \frac{-16\alpha\varepsilon}{1+\alpha^2}+\mathrm{O}(\varepsilon^2),\\
\frac{f_0'''}{f_0} &= \frac{-2\mathrm{i}\alpha(1+\alpha^2-3t_0(1-t_0))}{t_0^3(1-t_0)^3} = \frac{-32\mathrm{i}\alpha}{(1+\alpha^2)^2}+\mathrm{O}(\varepsilon).
\end{aligned} \tag{A.9}$$

We also expand $g(t)$, and perform the limit $\varepsilon \to 0$ in this slowly varying factor,

$$g_0 = \frac{(1+\mathrm{i}\alpha)^2}{4}\left(\frac{1+\mathrm{i}\alpha}{1-\mathrm{i}\alpha}\right)^{1-k(1+\mathrm{i}\alpha)}, \qquad \frac{g_0'}{g_0} = \frac{4\mathrm{i}\alpha-8}{1+\alpha^2}. \tag{A.10}$$

Only the zeroth order, g_0, turns out to be significant, but g_0' is needed for the error estimate. Later we will also study certain linear combinations where the first order gives the dominant contribution, since the zeroth order of the slowly varying factor vanishes for $\varepsilon \to 0$, cf (A.19).

We introduce a new integration variable, $t = x+\xi$, with $\xi = (\mathrm{i}\varepsilon/2)(1+\alpha^2)$, to remove the quadratic term in $\varphi(t)$, cf (A.8),

$$n\varphi(t) = \frac{4}{3}n\alpha(1+\alpha^2)\varepsilon^3 - 4\mathrm{i}\,n\alpha\varepsilon^2 x - \frac{16\mathrm{i}n\alpha x^3}{3(1+\alpha^2)^2}. \tag{A.11}$$

The saddle point t_0 lies in the lower half-plane, and the integration path in (A.1) is taken anticlockwise. Extending the path through the saddle point to infinity, we arrive at

$$F_{k-n} \approx \frac{3\mathrm{e}^{\pi\rho}f_0^n g_0}{\pi\rho^3}\int_{-\infty}^{+\infty}\mathrm{e}^{n\varphi(x+\xi)}\left(1+\mathrm{O}\left(\frac{g_0'}{g_0}x\right)\right)\mathrm{d}x. \tag{A.12}$$

This can be reduced to Airy functions, by virtue of

$$\int_{-\infty}^{+\infty}\exp(-\mathrm{i}(ax+bx^3))\,\mathrm{d}x = \frac{2\pi}{(3b)^{1/3}}\mathrm{Ai}\left(\frac{a}{(3b)^{1/3}}\right) \tag{A.13}$$

and its a-derivative (needed for the error term). Analytic continuation is implied, both in a and b, choosing $(-b)^{1/3} = -b^{1/3}$ if necessary. (One may view this as a distribution, a limit definition, $b \to 0$, of the Dirac function $2\pi\delta(a)$, though we use it with finite b.) Hence,

$$F_{k-n} \approx \frac{3\mathrm{e}^{\pi\rho}(1+\alpha^2)^{2/3}}{(2\alpha n)^{1/3}\rho^3} g_0\, \mathrm{e}^{n\log f_0}[\mathrm{Ai}(\lambda) + \mathrm{O}(\lambda \mathrm{Ai}'(\lambda))], \tag{A.14}$$

$$\lambda := (1+\alpha^2)^{2/3}(2\alpha n)^{2/3}\varepsilon^2 = \frac{2^{2/3}(\alpha k + 2\mathrm{i})}{(n\alpha)^{1/3}(1+\alpha^2)^{1/3}}, \tag{A.15}$$

with α and ε defined in (A.3). This expansion applies in the limit $n, \rho \to \infty$, with the fixed but otherwise arbitrary ratio $\alpha_\infty = \rho/n$. Finally, we note the ascending series of the Airy function

$$\mathrm{Ai}(\lambda) = \frac{1+\lambda^3/6}{3^{2/3}\Gamma(2/3)} - \frac{\lambda}{3^{1/3}\Gamma(1/3)} + \mathrm{O}(\lambda^4), \tag{A.16}$$

and conclude that only the zeroth order of $\mathrm{Ai}(\lambda)$ is significant in (A.14), so that we can replace the Airy function and the error term by $(3^{2/3}\Gamma(2/3))^{-1} + \mathrm{O}(\lambda)$.

Remark. There is another way to derive (A.14), based on the opposite identification made in (A.5). That is, we interchange the first two parameters in F_{k-n}, by choosing $b = 2 - \mathrm{i}\rho$ and $a = k - n$ in (A.1). For negative integer a, one of the Γ-functions in (A.1) gets singular, but this can be settled by replacing $k \to k + \delta$ and expanding in δ, to make the coefficient in front of the loop integral well defined. The integrand is not affected by that. We thus find, as a counterpart to (A.5), the representation

$$F_{k-n} = \frac{3\mathrm{i}(-)^{n-k+1}\oint t^{k-n-1}(1-t)^{3-k+n}(1-tz)^{-2+\mathrm{i}\rho}\,\mathrm{d}t}{\pi(n-k+3)(n-k+2)(n-k+1)}. \tag{A.17}$$

The integration loop is again anticlockwise encircling the interval [0, 1], with $1/z$ excluded. We write $k - n = (2 - \mathrm{i}\rho)/(\mathrm{i}\alpha)$, and factorize the integrand as $\mathrm{e}^{\mathrm{i}\rho\log f(t)}g(t)$, where

$$f(t) = t^{-1/(\mathrm{i}\alpha)}(1-t)^{1/(\mathrm{i}\alpha)}(1-tz), \qquad g(t) = t^{-1+2/(\mathrm{i}\alpha)}(1-t)^{3-2/(\mathrm{i}\alpha)}(1-tz)^{-2}. \tag{A.18}$$

This is an alternative starting point for the steepest descent, equivalent to (A.5) and (A.6) as long as α_∞ is kept constant, which can be used as a consistency check.

We turn to the hypergeometric aggregates in section 4,

$$H_{k,n} := F_{k-n} - \frac{1+\mathrm{i}\alpha_\infty}{1-\mathrm{i}\alpha_\infty}F_{k+1-n}, \tag{A.19}$$

which arise in the leading order of the $1/n$-expansion of (4.18) and (4.20), cf (A.27). To relate this to the foregoing discussion, we consider F_{l-n}, cf (A.2), with integer l, and note that the argument z in (A.2) does not depend on k or l, nor does the ε-expansion of z in (A.4), even though ε depends on k. We split the integrand defining F_{l-n} according to (A.5) ($k \to l$) into a fast and slowly varying part, $\mathrm{e}^{n\log f(t)}g(t)(1-tz)^{k-l}$, where $f(t)$ and $g(t)$ are defined in (A.6). Identifying $l = k + 1$, we find

$$H_{k,n} \approx \frac{3\mathrm{e}^{\pi\rho}}{\pi\rho^3}\oint \mathrm{e}^{n\log f(t)}h(t)\,\mathrm{d}t, \qquad h(t) := \left(1 - \frac{1+\mathrm{i}\alpha_\infty}{1-\mathrm{i}\alpha_\infty}\frac{1}{1-tz}\right)g(t). \tag{A.20}$$

The saddle point formulae derived in (A.7)–(A.9), (A.11) and (A.13) remain unchanged, and (A.10) is replaced by

$$h_0 := h(t_0) \approx 2\varepsilon\alpha_\infty g_0, \qquad h_0' \approx \frac{4\mathrm{i}\alpha_\infty}{1+\alpha_\infty^2}g_0 \tag{A.21}$$

with g_0 in (A.10). We thus find, analogously to (A.14),

$$H_{k,n} \approx \frac{3e^{\pi\rho}\left(1+\alpha_\infty^2\right)^{2/3}}{(2\alpha_\infty n)^{1/3}\rho^3} g_0\, e^{n \log f_0} \times \left[\frac{-2^{2/3}\alpha_\infty}{\left(1+\alpha_\infty^2\right)^{1/3}} \frac{\mathrm{Ai}'(\lambda)}{(\alpha_\infty n)^{1/3}} + 2\varepsilon\alpha_\infty \mathrm{Ai}(\lambda) + \mathrm{O}\left(\frac{\alpha_\infty\,|\varepsilon\lambda|}{|\alpha_\infty k + 2\mathrm{i}|}\right)\right], \qquad \text{(A.22)}$$

where λ is defined in (A.15) and $\log f_0$ in (A.9). Here, we may approximate, cf (A.16) and the error term in (A.22), $\mathrm{Ai}'(\lambda) \approx -(3^{1/3}\Gamma(1/3))^{-1}$ and $\mathrm{Ai}(\lambda) \approx (3^{2/3}\Gamma(2/3))^{-1}$.

The asymptotic formula (A.14) for F_{k-n} as well as (A.22) can be further simplified by systematically expanding in $1/n$. We start with (A.3),

$$\alpha = \alpha_\infty(1 + \kappa/n + \mathrm{O}(n^{-2})), \qquad \kappa := k + 2\mathrm{i}/\alpha_\infty,$$
$$\varepsilon = \kappa^{1/2}\left(1+\alpha_\infty^2\right)^{-1/2} n^{-1/2}(1 + \mathrm{O}(\kappa/n)), \qquad \text{(A.23)}$$

and note the $1/n$-expansion of $\log f_0$ in (A.9),

$$\log f_0 = \left(1 + \mathrm{i}\alpha_\infty + \frac{\mathrm{i}\alpha_\infty\kappa}{n}\right) \log\frac{1+\mathrm{i}\alpha_\infty}{1-\mathrm{i}\alpha_\infty} + \mathrm{O}(n^{-2}). \qquad \text{(A.24)}$$

Since α_∞ is positive, we can identify the logarithm with $2\mathrm{i}\arctan\alpha_\infty$. As for g_0 in (A.10), we may there simply replace α by α_∞, up to terms of $\mathrm{O}(1/n)$. The final form of (A.14) is thus:

$$F_{k-n} \approx \frac{3^{1/3}\left(1+\alpha_\infty^2\right)^{2/3}(1+\mathrm{i}\alpha_\infty)^2}{2^{7/3}\Gamma(2/3)(\alpha_\infty n)^{10/3}} \exp(\pi n\alpha_\infty) \times \exp[2\mathrm{i}(n(1+\mathrm{i}\alpha_\infty) - (k+1))\arctan\alpha_\infty](1 + \mathrm{O}(\lambda)), \qquad \text{(A.25)}$$

where $\lambda \propto n^{-1/3}$ defines the error term, cf (A.15). The analogous expansion of the aggregates (A.22) reads

$$H_{k,n} \approx \frac{3^{2/3}\alpha_\infty\left(1+\alpha_\infty^2\right)^{1/3}(1+\mathrm{i}\alpha_\infty)^2}{2^{5/3}\Gamma(1/3)(\alpha_\infty n)^{11/3}} \exp(\pi n\alpha_\infty) \times \exp[2\mathrm{i}(n(1+\mathrm{i}\alpha_\infty) - (k+1))\arctan\alpha_\infty] \times \left(1 + \frac{2^{1/3}\Gamma(1/3)\sqrt{\alpha_\infty k + 2\mathrm{i}}}{3^{1/3}\Gamma(2/3)(\alpha_\infty n)^{1/6}\left(1+\alpha_\infty^2\right)^{1/6}} + \mathrm{O}(n^{-1/2}, \rho^{-1/2})\right), \qquad \text{(A.26)}$$

where $\Gamma(1/3) \approx 2.679$ and $\Gamma(2/3) \approx 1.354$.

Returning to the matrix elements $R^{\mathrm{T,L}}$ in (4.18) and (4.20), we find in leading, non-vanishing order, up to terms of $\mathrm{O}(1/n)$,

$$R^{\mathrm{T}} \sim 2n^2 A H_{1,n}, \qquad R^{\mathrm{L}} \sim \frac{4r_{\mathrm{B}}^2 n^2\rho^2 A H_{1,n}}{(1-\mathrm{i}\alpha_\infty)^2 y} = -\frac{4\alpha_\infty^2 r_{\mathrm{B}}^2 n^4}{1+\alpha_\infty^2} A H_{1,n}, \qquad \text{(A.27)}$$

with $H_{1,n}$ in (A.19). The shortcuts A and y are defined in (4.18). Finally, by virtue of (A.26),

$$A H_{1,n} \approx \frac{-k_{\mathrm{e}}^{1/2} r_{\mathrm{B}}\alpha_\infty^{4/3}}{3^{1/3}2^{1/6}\Gamma(1/3)\left(1+\alpha_\infty^2\right)^{2/3} n^{13/6}}(1 + \cdots), \qquad \text{(A.28)}$$

where the parentheses are to be completed as in (A.26), with $k = 1$ and $\alpha_\infty = \rho/n$. The expansion of the squared matrix elements is stated in (4.25) and (4.26).

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